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ON THE COMPATIBILITY OF THE BETTI HARMONIC COPRODUCT WITH CYCLOTOMIC FILTRATIONS

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Abstract. In [Yad2], the second author introduced a Betti counterpart of N-cyclotomic double shuffle theory for any $N \geq 1$. The construction is based on the group algebra of the free group F_2, endowed with a filtration relative to a morphism $F_2 \to \mu_N$ (where μ_N is the group of N-th roots of unity). One of the main results of [Yad2] is the construction of a complete Hopf algebra coproduct $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ on the relative completion of a specific subalgebra $\mathcal{W}^{\mathrm{B}}$ of the group algebra of F_2. However, an explicit formula for this coproduct is missing. In this paper, we show that the discrete Betti harmonic coproduct $\Delta^{\mathcal{W},\mathrm{B}}$ defined in [EF1] for the classical case ($N = 1$) by the first author and Furusho remains compatible with the filtration structure on $\mathcal{W}^{\mathrm{B}}$ induced by the relative completion for arbitrary N. This compatibility suggests that the completion corresponding to $\Delta^{\mathcal{W},\mathrm{B}}$ is a candidate for an explicit realization of $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$.

Contents

1. Introduction

Throughout this paper, let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra and N be a positive integer. Denote by μ_N the group of complex N-th roots of unity with generator $\zeta_N := e^{\frac{\mathrm{i}2\pi}{N}}$. We will also use the following convention

Convention*. *For $\mathbf{k}$-submodules $A_1, \ldots, A_k$ of a $\mathbf{k}$-algebra A and positive integers $n_1, \ldots, n_k$, we denote by $A_1^{n_1} \cdots A_k^{n_k}$ the image of the morphism $A_1^{\otimes n_1} \otimes \cdots \otimes A_k^{\otimes n_k} \to A$ induced by the product in A. In the expression $A_1^{n_1} \cdots A_k^{n_k}$, we write A_j instead of $A_j^{n_j}$ whenever $n_i = 1$ $(1 \leq j \leq k)$.*

1.1. **Context and motivation.** Cyclotomic multiple zeta values (CMZVs) are special values of multiple polylogarithms evaluated at roots of unity, defined by the convergent series:

$$\mathrm{Li}_{(k_1,\ldots,k_r)}(z_1, \ldots, z_r) := \sum_{m_1>\cdots>m_r>0} \frac{z_1^{m_1} \cdots z_r^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}},$$

where $r, k_1, \ldots, k_r \in \mathbb{Z}_{>0}$ and $z_1, \ldots, z_r \in \mu_N$ with $(k_1, z_1) \neq (1, 1)$. These values arise as periods of the motivic fundamental groupoid of the cyclotomic punctured projective line $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ [Del10, Gon05] and are related to associators, mixed Tate motives, and the Grothendieck-Teichmüller group.

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From this perspective, the double shuffle relations among CMZVs –arising from series expansions and iterated integrals– are encoded in Racinet's "*double mélange et régularisation* scheme" DMR_N [Rac02], which is expressed in terms of a graded algebra $\mathcal{V}_N^{\mathrm{DR}}$, a graded subalgebra $\mathcal{W}_N^{\mathrm{DR}}$ of $\mathcal{V}_N^{\mathrm{DR}}$, and a Hopf algebra coproduct $\Delta_N^{\mathcal{W},\mathrm{DR}}$ on $\mathcal{W}_N^{\mathrm{DR}}$. More specifically, this framework is built on the completions of these graded objects, that is, the topological algebra $\widehat{\mathcal{V}}_N^{\mathrm{DR}}$ and the complete Hopf algebra $(\widehat{\mathcal{W}}_N^{\mathrm{DR}}, \widehat{\Delta}_N^{\mathcal{W},\mathrm{DR}})$ [Rac02, Yad1].

A Betti analogue of this setting was developed by the second author in [Yad2], generalizing the work of the first author and Furusho in [EF1] (for $N=1$), which in turn is inspired by the unpublished preprint of Deligne and Terasoma [DeT]. Here, the key objects are a filtered algebra $\mathcal{V}_N^{\mathrm{B}}$ and subalgebra $\mathcal{W}_N^{\mathrm{B}}$ of $\mathcal{V}_N^{\mathrm{B}}$; and the completion $\widehat{\mathcal{W}}_N^{\mathrm{B}}$, equipped with a complete Hopf algebra coproduct $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ –called the N-cyclotomic Betti harmonic coproduct– whose defining property is the conjugation formula [Yad2, Theorem 3.2.4]

$$\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}} = (\mathrm{comp}_{\Phi,N}^{\mathcal{W}} \otimes \mathrm{comp}_{\Phi,N}^{\mathcal{W}})^{-1} \circ \widehat{\Delta}_N^{\mathcal{W},\mathrm{DR}} \circ \mathrm{comp}_{\Phi,N}^{\mathcal{W}}, \tag{1}$$

which is valid for any choice of $\Phi \in \mathsf{DMR}_N$; where $\mathrm{comp}_{\Phi,N}^{\mathcal{W}} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_N^{\mathrm{DR}}$ is a comparison isomorphism [Yad2, Proposition-Definition 3.2.2] attached to Φ.

For $N=1$, a Hopf algebra coproduct $\Delta^{\mathcal{W},\mathrm{B}}$ on $\mathcal{W}_1^{\mathrm{B}} = \mathcal{W}^{\mathrm{B}}$ was explicitly constructed in [EF1, EF2], the compatibilty of $\Delta^{\mathcal{W},\mathrm{B}}$ with the filtration on $\mathcal{W}^{\mathrm{B}}$ for $N=1$ was proved, and the corresponding completed coproduct $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}$ was identified with $\widehat{\Delta}_1^{\mathcal{W},\mathrm{B}}$ from (1), hence $\widehat{\Delta}_1^{\mathcal{W},\mathrm{B}} = \widehat{\Delta}^{\mathcal{W},\mathrm{B}}$. However, for general N, an explicit formula for $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ is still unknown.

1.2. **The main results.** Let F_2 be the free group generated by two elements denoted X_0 and X_1. Consider the group morphism $F_2 \to \mu_N$ given by

$$X_0 \mapsto \zeta_N \text{ and } X_1 \mapsto 1.$$

Its kernel is the group freely generated by the $N+1$ elements [Yad2, Lemma 3.1.1]

$$X_0^N \text{ and } X_0^a X_1 X_0^{-a}, \text{ for } a \in [\![0, N-1]\!].$$

Denote by $\mathcal{I}_N := \ker(\mathbf{k}F_2 \to \mathbf{k}\mu_N)$ where $\mathbf{k}F_2 \to \mathbf{k}\mu_N$ is the $\mathbf{k}$-algebra morphism induced from the group morphism $F_2 \to \mu_N$.

Definition 1.1 ([Yad2, Proposition-Definition 3.1.4])**.** Let $\mathcal{V}_N^{\mathrm{B}}$ be the group algebra $\mathbf{k}F_2$ equipped with the algebra filtration given by

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} := \begin{cases} \mathbf{k}F_2 & \text{if } m \leq 0 \\ \mathcal{I}_N^m & \text{if } m > 0 \end{cases},$$

where $\mathcal{I}_N^m$ is the m-th power of the ideal $\mathcal{I}_N$ (see Convention*).

Definition 1.2 ([Yad2, Proposition-Definition 3.1.13])**.** Consider the subalgebra $\mathcal{W}_N^{\mathrm{B}}$ of $\mathcal{V}_N^{\mathrm{B}}$ given by

$$\mathcal{W}_N^{\mathrm{B}} := \mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1).$$

It is endowed with the algebra filtration given by

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} := \mathcal{W}_N^{\mathrm{B}} \cap \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}, \ \forall m \in \mathbb{Z}.$$

When $N=1$, the filtration $(\mathcal{F}^m\mathcal{V}_1^{\mathrm{B}})_{m\in\mathbb{Z}}$ is the natural filtration of the group algebra $\mathbf{k}F_2$ given by powers of the augmentation ideal. Therefore, the induced filtration on $\mathcal{W}_1^{\mathrm{B}}$ corresponds the one given in [EF1, Sec. 2.1]. We will use the notation $\mathcal{V}^{\mathrm{B}}$ (resp. $\mathcal{W}^{\mathrm{B}}$) instead of $\mathcal{V}_1^{\mathrm{B}}$ (resp. $\mathcal{W}_1^{\mathrm{B}}$) to refer to these naturally filtered algebras.

It follows from [EF1, Proposition 2.3] that the algebra $\mathcal{W}^{\mathrm{B}}$ is generated by

$$X_1^{-1} \text{ and } X_0^n(X_1-1) \text{ for } n\in\mathbb{Z}.$$

The algebra $\mathcal{W}^{\mathrm{B}}$ is equipped with a bialgebra structure whose coproduct is the algebra morphism $\Delta^{\mathcal{W},\mathrm{B}}:\mathcal{W}^{\mathrm{B}}\to\mathcal{W}^{\mathrm{B}}\otimes\mathcal{W}^{\mathrm{B}}$ given by (see [EF1, Lemma 2.11])

$$\Delta^{\mathcal{W},\mathrm{B}}(X_1^{-1})=X_1^{-1}Y_1^{-1},$$

and for $n\in\mathbb{Z}$,

$$\Delta^{\mathcal{W},\mathrm{B}}(X_0^n(X_1-1))=X_0^n(X_1-1)+Y_0^n(Y_1-1)-\sum_{k=1}^{n-1}X_0^k(X_1-1)Y_0^{n-k}(Y_1-1),$$

where one sets $X_i^{\pm1}:=X_i^{\pm1}\otimes 1$ and $Y_i^{\pm1}:=1\otimes X_i^{\pm1}$ for $i\in\{0,1\}$, and one uses the convention that for a map f from $\mathbb{Z}$ to an abelian group and $p,q\in\mathbb{Z}$,

$$\sum_{k=p}^{q}f(k):=\begin{cases}f(p)+\cdots+f(q) & \text{if } q>p-1\\ 0 & \text{if } q=p-1\\ -f(p-1)-\cdots-f(q+1) & \text{if } q<p-1\end{cases}$$

The following result is the first main theorem of the paper. It states that the coproduct $\Delta^{\mathcal{W},\mathrm{B}}$ is actually compatible with the filtration given in Definition 1.2:

Theorem 1.3. *For any $m\in\mathbb{Z}$, we have*

$$\Delta^{\mathcal{W},\mathrm{B}}(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})\subset\mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}}).$$

Definition 1.4 ([Yad1, §2.1.1])**.** Let $\mathcal{V}_N^{\mathrm{DR}}$ be the graded $\mathbf{k}$-algebra[1] generated by $\{e_0,e_1\}\sqcup\mu_N$ where e_0 and e_1 are of degree 1 and elements $\zeta\in\mu_N$ are of degree 0 satisfying the relations:

(i) $\zeta\cdot\eta=\zeta\eta$; (ii) $1_{\mathcal{V}_N^{\mathrm{DR}}}=1$; (iii) $\zeta\cdot e_0=e_0\cdot\zeta$;

for any $\zeta,\eta\in\mu_N$; where "$\cdot$" is the algebra multiplication[2].

Recall from [Yad1, §2.1.1] the subalgebra

$$\mathcal{W}_N^{\mathrm{DR}}:=\mathbf{k}\oplus\mathcal{V}_N^{\mathrm{DR}}e_1$$

of $\mathcal{V}_N^{\mathrm{DR}}$. It is a graded algebra freely generated by ([Yad1, Proposition 2.6(ii)])

$$Z:=\{z_{n,\zeta}:=-e_0^{n-1}\zeta e_1\,|\,(n,\zeta)\in\mathbb{Z}_{>0}\times\mu_N\},$$

where for any $(n,\zeta)\in\mathbb{Z}_{>0}\times\mu_N$ the element $z_{n,\zeta}$ is of degree n. Moreover, $\mathcal{W}_N^{\mathrm{DR}}$ is equipped with a Hopf algebra structure with respect to the *harmonic coproduct*, which is the algebra morphism $\Delta_N^{\mathcal{W},\mathrm{DR}}:\mathcal{W}_N^{\mathrm{DR}}\to\mathcal{W}_N^{\mathrm{DR}}\otimes\mathcal{W}_N^{\mathrm{DR}}$ given by ([Yad1, Proposition 2.11(i)])

$$\Delta_N^{\mathcal{W},\mathrm{DR}}(z_{n,\zeta})=z_{n,\zeta}\otimes 1+1\otimes z_{n,\zeta}+\sum_{\substack{k=1\\ \eta\in\mu_N}}^{n-1}z_{k,\eta}\otimes z_{n-k,\zeta\eta^{-1}}.$$

Let $\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ be the associated graded algebra of $\mathcal{V}_N^{\mathrm{B}}$ for the μ_N-filtration $(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{Z}}$. For $m\in\mathbb{Z}$ and $v\in\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$, denote by $[v]_m$ the image in $\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{V}_N^{\mathrm{B}}$ of the element v.

Proposition 1.5 ([Yad, Theorem 3.1.6] and [Yad, Proposition 3.1.12])**.**

(a) There exists a graded algebra isomorphism $\rho_N^{\mathcal{V}}:\mathcal{V}_N^{\mathrm{DR}}\to\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ uniquely defined by

$$\zeta_N\mapsto[X_0]_0,\quad e_0\mapsto[X_0^N-1]_1,\quad e_1\mapsto[X_1-1]_1.$$

[1] in [Yad1, §2.1.1] this corresponds to $\mathcal{V}_G$ for $G=\mu_N$.

[2] which we will omit if there is no risk of ambiguity.

(b) The graded algebra isomorphism $\rho_N^{\mathcal{V}} : \mathcal{V}_N^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ restricts to a graded algebra isomorphism $\rho_N^{\mathcal{W}} : \mathcal{W}_N^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}})$.

By Theorem 1.3, the filtered algebra morphism $\Delta^{\mathcal{W},\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}}$ induces the graded algebra morphism

$$\mathrm{gr}(\Delta^{\mathcal{W},\mathrm{B}}) : \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) \to \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) \otimes \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}).$$

The following result is the second main theorem of the paper. It states that the associated graded algebra morphism $\mathrm{gr}(\Delta^{\mathcal{W},\mathrm{B}})$ is in fact the graded algebra morphism $\Delta_N^{\mathcal{W},\mathrm{DR}}$.

Theorem 1.6. *The following diagram*

$$\begin{array}{ccc} \mathcal{W}_N^{\mathrm{DR}} & \xrightarrow{\Delta^{\mathcal{W},\mathrm{DR}}} & \mathcal{W}_N^{\mathrm{DR}} \otimes \mathcal{W}_N^{\mathrm{DR}} \\ {\scriptstyle \rho_N^{\mathcal{W}}}\downarrow & & \downarrow{\scriptstyle \rho_N^{\mathcal{W}} \otimes \rho_N^{\mathcal{W}}} \\ \mathrm{gr}(\mathcal{W}^{\mathrm{B}}) & \xrightarrow{\mathrm{gr}(\Delta^{\mathcal{W},\mathrm{B}})} & \mathrm{gr}(\mathcal{W}^{\mathrm{B}}) \otimes \mathrm{gr}(\mathcal{W}^{\mathrm{B}}) \end{array} \tag{2}$$

commutes.

Finally, regarding the topological algebra morphism $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ given in (1), Theorems 1.3 and 1.6 motivate the following problem:

Problem 1.7. *For suitable $a, b \in \mathbb{Z}$, show that the topological algebra morphism $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ is the completion (w.r.t. the filtration $(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})_{m\in\mathbb{Z}}$) of the algebra morphism $\mathrm{Ad}_{X_1^a Y_1^b} \circ \Delta^{\mathcal{W},\mathrm{B}}$.*

2. Compatibility of $\Delta^{\mathcal{W},\mathrm{B}}$ with the filtration $(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{Z}}$

In this section, we prove Theorem 1.3. To do so, we will start with some preparatory results.

Lemma 2.1. *For $m \in \mathbb{Z}_{>0}$, we have*

(a) $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$. *(b) $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$.*

(c) $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}$ is a left $\mathcal{V}_N^{\mathrm{B}}$-module.

Proof. For (a) and (b), see [Yad2, Lemma 3.1.14]. (c) follows immediately from (b). □

Lemma 2.2. *For $m \in \mathbb{Z}$, we have*

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \begin{cases} \mathcal{W}_N^{\mathrm{B}} & \text{if } m \leq 0 \\ \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) & \text{if } m = 1 \\ (X_0^N - 1)^{m-1}\mathbf{k}[X_0, X_0^{-1}](X_1 - 1) + \sum_{k=1}^{m-1} \mathcal{F}^k\mathcal{W}_N^{\mathrm{B}} \cdot \mathcal{F}^{m-k}\mathcal{W}_N^{\mathrm{B}} & \text{if } m \geq 2 \end{cases}$$

Proof. The result is immediate for $m = 0$; and for $m = 1$, it follows from Lemma 2.1 (b). We now consider the case $m \geq 2$. Since $(\mathcal{F}^n\mathcal{W}_N^{\mathrm{B}})_{n\in\mathbb{Z}}$ is a decreasing algebra filtration, then

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \supset \sum_{k=1}^{m-1} \mathcal{F}^k\mathcal{W}_N^{\mathrm{B}} \cdot \mathcal{F}^{m-k}\mathcal{W}_N^{\mathrm{B}}. \tag{3}$$

On the other hand, since $X_0^N - 1$ and $X_1 - 1$ belong to $\mathcal{I}_N$, we obtain the inclusion in the following

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \supset (X_0^N - 1)^{m-1}\mathbf{k}[X_0, X_0^{-1}](X_1 - 1), \tag{4}$$

and the equality follows from Lemma 2.1 (a). From (3) and (4), we obtain the following inclusion

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \supset (X_0^N-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X_1-1)+\sum_{k=1}^{m-1}\mathcal{F}^k\mathcal{W}_N^{\mathrm{B}}\cdot\mathcal{F}^{m-k}\mathcal{W}_N^{\mathrm{B}}.$$

Let us now prove the converse. The group morphism $F_2\to\mathbb{Z}$ given by $X_0\mapsto 1$ and $X_1\mapsto 0$ admits a section given by $1\mapsto X_0$. Then $\mathbf{k}F_2$ is the direct sum of the image of the section $\mathbf{k}\mathbb{Z}\to\mathbf{k}F_2$, which is $\mathbf{k}[X_0,X_0^{-1}]$, and of the kernel of $\mathbf{k}F_2\to\mathbf{k}\mathbb{Z}$, which is the two-sided ideal of $\mathbf{k}F_2$ generated by X_1-1. Let us denote by $\mathcal{V}_N^{\mathrm{B}}(X_1-1)\mathcal{V}_N^{\mathrm{B}}$ this ideal[3].
We derive the direct sum decomposition[4]

$$\mathcal{V}_N^{\mathrm{B}}=\mathbf{k}[X_0,X_0^{-1}]\oplus\mathcal{V}_N^{\mathrm{B}}(X_1-1)\mathcal{V}_N^{\mathrm{B}}.$$

Moreover, since $\mathcal{V}_N^{\mathrm{B}}(X_1-1)\mathcal{V}_N^{\mathrm{B}}\subset\mathcal{I}_N=\ker(\mathbf{k}F_2\to\mathbf{k}\mu_N)$, we have

$$\mathcal{I}_N=\ker\big(\mathbf{k}[X_0,X_0^{-1}]\to\mathbf{k}\mu_N\big)\oplus\mathcal{V}_N^{\mathrm{B}}(X_1-1)\mathcal{V}_N^{\mathrm{B}},$$

where $\mathbf{k}[X_0,X_0^{-1}]\to\mathbf{k}\mu_N$ is the restriction of $\mathbf{k}F_2\to\mathbf{k}\mu_N$ to $\mathbf{k}[X_0,X_0^{-1}]$. Therefore,

$$\mathcal{I}_N=(X_0^N-1)\mathbf{k}[X_0,X_0^{-1}]\oplus\mathcal{V}_N^{\mathrm{B}}(X_1-1)\mathcal{V}_N^{\mathrm{B}}. \tag{5}$$

Denote by $\mathcal{A}_0=(X_0^N-1)\mathbf{k}[X_0,X_0^{-1}]$ and $\mathcal{A}_1=\mathcal{V}_N^{\mathrm{B}}(X_1-1)\mathcal{V}_N^{\mathrm{B}}$. Thanks to (5), we obtain

$$\mathcal{I}_N^{m-1}=\sum_{\lambda:[\![1,m-1]\!]\to\{0,1\}}\mathcal{A}_{\lambda(1)}\cdots\mathcal{A}_{\lambda(m-1)}=\mathcal{A}_0^{m-1}+\sum_{\substack{\lambda:[\![1,m-1]\!]\to\{0,1\}\\ \lambda\neq\mathbf{0}}}\mathcal{A}_{\lambda(1)}\cdots\mathcal{A}_{\lambda(m-1)}, \tag{6}$$

where $\mathbf{0}:[\![1,m-1]\!]\to\{0,1\}$ is the zero map.
Set $X(0):=X_0^N$ and $X(1):=X_1$. Since $\mathcal{A}_i\subset\mathcal{V}_N^{\mathrm{B}}(X(i)-1)\mathcal{V}_N^{\mathrm{B}}$ (for $i\in\{0,1\}$), it follows that for any map $\lambda:[\![1,m-1]\!]\to\{0,1\}$, we have

$$\mathcal{A}_{\lambda(1)}\cdots\mathcal{A}_{\lambda(m-1)}\subset\mathcal{V}_N^{\mathrm{B}}(X(\lambda(1))-1)\mathcal{V}_N^{\mathrm{B}}\cdots\mathcal{V}_N^{\mathrm{B}}(X(\lambda(m-1))-1)\mathcal{V}_N^{\mathrm{B}}. \tag{7}$$

Combining equality (6), inclusion (7) for $\lambda\neq\mathbf{0}$, and the equality $\mathcal{A}_0^{m-1}=(X_0^N-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}]$, we obtain

$$\mathcal{I}_N^{m-1}\subset(X(0)-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}]+\sum_{\substack{\lambda:[\![1,m-1]\!]\to\{0,1\}\\ \lambda\neq\mathbf{0}}}\mathcal{V}_N^{\mathrm{B}}(X(\lambda(1))-1)\mathcal{V}_N^{\mathrm{B}}\cdots\mathcal{V}_N^{\mathrm{B}}(X(\lambda(m-1))-1)\mathcal{V}_N^{\mathrm{B}}.$$

Since $X(i)-1\in\mathcal{I}_N$ (for $i\in\{0,1\}$), the right hand side of this inclusion is contained in $\mathcal{I}_N^{m-1}$, therefore

$$\mathcal{I}_N^{m-1}=(X(0)-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}]+\sum_{\substack{\lambda:[\![1,m-1]\!]\to\{0,1\}\\ \lambda\neq\mathbf{0}}}\mathcal{V}_N^{\mathrm{B}}(X(\lambda(1))-1)\mathcal{V}_N^{\mathrm{B}}\cdots\mathcal{V}_N^{\mathrm{B}}(X(\lambda(m-1))-1)\mathcal{V}_N^{\mathrm{B}}. \tag{8}$$

[3]recall that the algebras $\mathcal{V}_N^{\mathrm{B}}$ and $\mathbf{k}F_2$ are equal. In the sequel, we use the former rather that the latter notation for denoting the two-sided ideal generated by X_1-1.

[4]where the first summand is a subalgebra of and the second summand is a two-sided ideal

Finally,

$$\begin{aligned}
\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} =&\mathcal{I}_N^{m-1}(X(1)-1)\\
=&(X(0)-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X(1)-1)\\
&+\sum_{\substack{\lambda:[\![1,m-1]\!]\to\{0,1\}\\ \lambda\neq\mathbf{0}}}\mathcal{V}_N^{\mathrm{B}}(X(\lambda(1))-1)\cdots\mathcal{V}_N^{\mathrm{B}}(X(\lambda(m-1))-1)\mathcal{V}_N^{\mathrm{B}}(X(1)-1)\\
=&(X(0)-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X(1)-1)\\
&+\sum_{\lambda\in\Lambda_m}\mathcal{V}_N^{\mathrm{B}}(X(\lambda(1))-1)\cdots\mathcal{V}_N^{\mathrm{B}}(X(\lambda(m-1))-1)\mathcal{V}_N^{\mathrm{B}}(X(\lambda(m))-1)\\
=&(X(0)-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X(1)-1)\\
&+\sum_{j\geq 2}\sum_{(k_1,\ldots,k_j)\in\mathcal{K}_m^{(j)}}\big(\mathcal{V}_N^{\mathrm{B}}(X(0)-1)\big)^{k_1-1}\mathcal{V}_N^{\mathrm{B}}(X(1)-1)\big(\mathcal{V}_N^{\mathrm{B}}(X(0)-1)\big)^{k_2-k_1-1}\\
&\qquad\qquad\mathcal{V}_N^{\mathrm{B}}(X(1)-1)\cdots\big(\mathcal{V}_N^{\mathrm{B}}(X(0)-1)\big)^{k_j-k_{j-1}-1}\mathcal{V}_N^{\mathrm{B}}(X(1)-1)\\
\subset&(X(0)-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X(1)-1)\\
&+\sum_{j\geq 2}\sum_{(k_1,\ldots k_j)\in\mathcal{K}_m^{(j)}}\mathcal{F}^{k_1}\mathcal{W}_N^{\mathrm{B}}\cdot\mathcal{F}^{k_2-k_1}\mathcal{W}_N^{\mathrm{B}}\cdots\mathcal{F}^{k_j-k_{j-1}}\mathcal{W}_N^{\mathrm{B}}\\
\subset&(X(0)-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X(1)-1)+\sum_{k=1}^{m-1}\mathcal{F}^k\mathcal{W}_N^{\mathrm{B}}\cdot\mathcal{F}^{m-k}\mathcal{W}_N^{\mathrm{B}},
\end{aligned}$$

where the first equality follows from Lemma 2.1 (b) and the second one from (8). In the third equality one denotes

$$\Lambda_m:=\{\lambda:[\![1,m]\!]\to\{0,1\}\mid\lambda(m)=1,\ \lambda_{|[\![1,m-1]\!]}\neq\mathbf{0}\}$$

and the equality then follows immediately. In the fourth equality one denotes

$$\mathcal{K}_m^{(j)}:=\{(k_1,\ldots,k_j)\mid 1\leq k_1<\cdots<k_{j-1}<k_j=m\},$$

one also uses Convention* for the definition of $\big(\mathcal{V}_N^{\mathrm{B}}(X(0)-1)\big)^k$ (for any integer $k\geq 1$); and the equality is induced by the bijection

$$\Lambda_m\simeq\bigsqcup_{j\geq 2}\mathcal{K}_m^{(j)},\quad\lambda\mapsto\lambda^{-1}(\{0\}).$$

The first inclusion follows from the fact $(\mathcal{V}_N^{\mathrm{B}}(X(0)-1))^{k-1}\mathcal{V}_N^{\mathrm{B}}(X(1)-1)\subset\mathcal{F}^k\mathcal{W}_N^{\mathrm{B}}$ (for any integer $k\geq 1$); and the last inclusion from the fact that $(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})_{m\in\mathbb{Z}}$ is a decreasing filtration and therefore

$$\mathcal{F}^{k_2-k_1}\mathcal{W}_N^{\mathrm{B}}\cdot\cdots\cdot\mathcal{F}^{k_j-k_{j-1}}\mathcal{W}_N^{\mathrm{B}}\subset\mathcal{F}^{k_j-k_1}\mathcal{W}_N^{\mathrm{B}}=\mathcal{F}^{m-k_1}\mathcal{W}_N^{\mathrm{B}}.$$

□

Lemma 2.3. *For any integer $m\geq 2$, we have*

$$\Delta^{\mathcal{W},\mathrm{B}}\big((X_0^N-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X_1-1)\big)\subset\mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}})$$

Proof. Let $P(X_0, X_0^{-1}) \in \mathbf{k}[X_0, X_0^{-1}]$. We have

$$\begin{aligned}(9)\qquad &\Delta^{\mathcal{W},\mathrm{B}}\big((X_0^N-1)^{m-1}P(X_0,X_0^{-1})(X_1-1)\big)\\ &= (X_0^N-1)^{m-1}P(X_0,X_0^{-1})(X_1-1) + (Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})(Y_1-1)\\ &- \frac{(X_0^N-1)^{m-1}P(X_0,X_0^{-1})Y_0 - (Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})X_0}{X_0-Y_0}(X_1-1)(Y_1-1),\end{aligned}$$

where $\frac{(X_0^N-1)^{m-1}P(X_0,X_0^{-1})Y_0-(Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})X_0}{X_0-Y_0}$ is the polynomial $F(X_0, X_0^{-1}, Y_0, Y_0^{-1}) \in \mathbf{k}[X_0, X_0^{-1}, Y_0, Y_0^{-1}]$ such that

$$(X_0-Y_0)F(X_0,X_0^{-1},Y_0,Y_0^{-1}) = (X_0^N-1)^{m-1}P(X_0,X_0^{-1})Y_0 - (Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})X_0.$$

Next, we have

$$\begin{aligned}&\frac{(X_0^N-1)^{m-1}P(X_0,X_0^{-1})Y_0 - (Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})X_0}{X_0-Y_0} = -(X_0^N-1)^{m-1}P(X_0,X_0^{-1})\\ &-(Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1}) + \frac{(X_0^N-1)^{m-1}P(X_0,X_0^{-1})X_0 - (Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})Y_0}{X_0-Y_0}\\ &= -(X_0^N-1)^{m-1}P(X_0,X_0^{-1}) - (Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})\\ &+ \frac{(X_0^N-1)^{m-1}\big(P(X_0,X_0^{-1})X_0 - P(Y_0,Y_0^{-1})Y_0\big)}{X_0-Y_0} + \frac{\big((X_0^N-1)^{m-1}-(Y_0^N-1)^{m-1}\big)P(Y_0,Y_0^{-1})Y_0}{X_0-Y_0}\end{aligned}$$

Denote by

$$A(X_0,X_0^{-1},Y_0,Y_0^{-1}) := -(X_0^N-1)^{m-1}P(X_0,X_0^{-1}) - (Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1}),$$

$$B(X_0,X_0^{-1},Y_0,Y_0^{-1}) := \frac{(X_0^N-1)^{m-1}\big(P(X_0,X_0^{-1})X_0 - P(Y_0,Y_0^{-1})Y_0\big)}{X_0-Y_0},$$

$$C(X_0,X_0^{-1},Y_0,Y_0^{-1}) := \frac{\big((X_0^N-1)^{m-1}-(Y_0^N-1)^{m-1}\big)P(Y_0,Y_0^{-1})Y_0}{X_0-Y_0}.$$

Thanks to this, we obtain from equality (9) the following identity

$$\begin{aligned}(10)\qquad &\Delta^{\mathcal{W},\mathrm{B}}\big((X_0^N-1)^{m-1}P(X_0,X_0^{-1})(X_1-1)\big) = (X_0^N-1)^{m-1}P(X_0,X_0^{-1})(X_1-1)\\ &+(Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})(Y_1-1) - A(X_0,X_0^{-1},Y_0,Y_0^{-1})(X_1-1)(Y_1-1)\\ &- B(X_0,X_0^{-1},Y_0,Y_0^{-1})(X_1-1)(Y_1-1) - C(X_0,X_0^{-1},Y_0,Y_0^{-1})(X_1-1)(Y_1-1).\end{aligned}$$

Since $X_0^N-1, X_1-1 \in \mathcal{I}_N$, we have

$$(11)\qquad (X_0^N-1)^{m-1}P(X_0,X_0^{-1})(X_1-1) \in \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \mathcal{W}^{\mathrm{B}} = \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}},$$

Then the statement (11) implies that

$$(X_0^N-1)^{m-1}P(X_0,X_0^{-1})(X_1-1) \in \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \otimes 1 \subset \mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}}),$$

and

$$(Y_0^N-1)^{m-1}P(Y_0,Y_0^{-1})(Y_1-1) \in 1 \otimes \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}}).$$

On the other hand, we have

$$\begin{aligned}(12)\qquad A(X_0,X_0^{-1},Y_0,Y_0^{-1})(X_1-1)(Y_1-1) &= -P(X_0,X_0^{-1})(X_0^N-1)^{m-1}(X_1-1)(Y_1-1)\\ &- P(Y_0,Y_0^{-1})(Y_0^N-1)^{m-1}(Y_1-1)(X_1-1)\\ &\in \mathcal{F}^{m+1}\left(\mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}}\right),\end{aligned}$$

where the "$\in$" claim follows from the fact that $(X_0^N-1)^{m-1}(X_1-1)(Y_1-1)\in\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{F}^1\mathcal{W}^{\mathrm{B}}$ and that $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{F}^1\mathcal{W}^{\mathrm{B}}$ is a left $(\mathcal{V}_N^{\mathrm{B}}\otimes\mathcal{V}_N^{\mathrm{B}})$-module, which implies

$$-P(X_0,X_0^{-1})(X_0^N-1)^{m-1}(X_1-1)(Y_1-1)\in\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{F}^1\mathcal{W}^{\mathrm{B}}.$$

Swapping between X and Y enables us to apply the same argument to show that

$$-P(Y_0,Y_0^{-1})(Y_0^N-1)^{m-1}(Y_1-1)(X_1-1)\in\mathcal{F}^1\mathcal{W}^{\mathrm{B}}\otimes\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}.$$

Moreover, we have

$$\begin{aligned}(13)\qquad & B(X_0,X_0^{-1},Y_0,Y_0^{-1})(X_1-1)(Y_1-1)\\ &=\frac{P(X_0,X_0^{-1})X_0-P(Y_0,Y_0^{-1})Y_0}{X_0-Y_0}(X_0^N-1)^{m-1}(X_1-1)(Y_1-1)\\ &\in\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}\subset\mathcal{F}^{m+1}\left(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}}\right),\end{aligned}$$

where the "$\in$" claim follows from the fact that $(X_0^N-1)^{m-1}(X_1-1)(Y_1-1)\in\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{F}^1\mathcal{W}^{\mathrm{B}}$ and that $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{F}^1\mathcal{W}^{\mathrm{B}}$ is a left $(\mathcal{V}_N^{\mathrm{B}}\otimes\mathcal{V}_N^{\mathrm{B}})$-module.
Moreover, we have

$$\begin{aligned}(14)\qquad & C(X_0,X_0^{-1},Y_0,Y_0^{-1})(X_1-1)(Y_1-1)\\ &=P(Y_0,Y_0^{-1})Y_0\frac{X_0^N-Y_0^N}{X_0-Y_0}\frac{(X_0^N-1)^{m-1}-(Y_0^N-1)^{m-1}}{X_0^N-Y_0^N}(X_1-1)(Y_1-1)\\ &=P(Y_0,Y_0^{-1})Y_0\left(\sum_{k=0}^{N-1}X_0^kY_0^{N-1-k}\right)\sum_{l=0}^{m-2}\underbrace{(X_0^N-1)^l(X_1-1)}_{\in\mathcal{F}^{l+1}\mathcal{W}_N^{\mathrm{B}}\otimes 1}\underbrace{(Y_0^N-1)^{m-2-l}(Y_1-1)}_{\in 1\otimes\mathcal{F}^{m-1-l}\mathcal{W}_N^{\mathrm{B}}}\\ &\in\mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}}).\end{aligned}$$

Therefore, it follows from identity (10) that

$$\Delta^{\mathcal{W},\mathrm{B}}\big((X_0^N-1)^{m-1}P(X_0,X_0^{-1})(X_1-1)\big)\in\mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}}).$$

□

Proof of Theorem 1.3. If $m\leq 0$, the result is immediate. Let us assume that $m\geq 1$. We will proceed with the proof by induction on m.
For $m=1$, denote by $\varepsilon:\mathcal{W}_N^{\mathrm{B}}\to\mathbf{k}$ the counit of the bialgebra $(\mathcal{W}_N^{\mathrm{B}},\Delta^{\mathcal{W},\mathrm{B}})$. We have

$$\Delta^{\mathcal{W},\mathrm{B}}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}})=\Delta^{\mathcal{W},\mathrm{B}}(\ker(\varepsilon))\subset\ker(\varepsilon\otimes\varepsilon)=\mathcal{F}^1(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}}),$$

where the first equality follows from the identity $\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}=\ker(\varepsilon)$; the second equality from the counit identity $\Delta^{\mathcal{W},\mathrm{B}}\circ\varepsilon=(\varepsilon\otimes\varepsilon)\circ\Delta^{\mathcal{W},\mathrm{B}}$; and the third equality from the identity $\mathcal{F}^1(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}})=\ker(\varepsilon\otimes\varepsilon)$.
Suppose now that the statement is true until $m-1$. We have

$$\begin{aligned}\Delta^{\mathcal{W},\mathrm{B}}(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})&=\Delta^{\mathcal{W},\mathrm{B}}\left((X_0^N-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X_1-1)+\sum_{k=1}^{m-1}\mathcal{F}^k\mathcal{W}_N^{\mathrm{B}}\cdot\mathcal{F}^{m-k}\mathcal{W}_N^{\mathrm{B}}\right)\\ &\subset\Delta^{\mathcal{W},\mathrm{B}}\left((X_0^N-1)^{m-1}\mathbf{k}[X_0,X_0^{-1}](X_1-1)\right)+\sum_{k=1}^{m-1}\Delta^{\mathcal{W},\mathrm{B}}\left(\mathcal{F}^k\mathcal{W}_N^{\mathrm{B}}\right)\cdot\Delta^{\mathcal{W},\mathrm{B}}\left(\mathcal{F}^{m-k}\mathcal{W}_N^{\mathrm{B}}\right)\\ &\subset\mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}})+\sum_{k=1}^{m-1}\mathcal{F}^k(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}})\cdot\mathcal{F}^{m-k}(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}})\\ &\subset\mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}}\otimes\mathcal{W}_N^{\mathrm{B}}),\end{aligned}$$

where the equality follows from Lemma 2.2, the first inclusion follows by linearity of $\Delta^{\mathcal{W},\mathrm{B}}$ and compatibility with the product; the second inclusion from Lemma 2.3 and induction hypothesis; and the last inclusion from the fact that $\big(\mathcal{F}^m(\mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}})\big)_{m\in\mathbb{Z}}$ is an algebra filtration, which follows from the fact that $(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})_{m\in\mathbb{Z}}$ is an algebra filtration. □

3. Computation of $\mathrm{gr}(\Delta^{\mathcal{W},\mathrm{B}})$

In this section, we prove Theorem 1.6.

Proof of Theorem 1.6. Let us prove that diagram (2) of graded algebra morphisms commutes for any degree $m \geq 1$.
For $a \in [\![0, N-1]\!]$, z_{m,ζ_N^a} is a degree m element of $\mathcal{W}_N^{\mathrm{DR}}$ and we have

$$\rho_N^{\mathcal{W}}(z_{m,\zeta_N^a}) = [(X_0^N - 1)^{m-1} X_0^a (1 - X_1)]_m. \tag{15}$$

Recall that

$$\Delta_N^{\mathcal{W},\mathrm{DR}}(z_{m,\zeta_N^a}) = z_{m,\zeta_N^a} \otimes 1 + 1 \otimes z_{m,\zeta_N^a} + \sum_{\substack{1\leq k\leq m-1\\0\leq b\leq N-1}} z_{k,\zeta_N^b} \otimes z_{m-k,\zeta_N^{a-b}}.$$

Therefore, we obtain

$$\begin{aligned}&\big(\rho_N^{\mathcal{W}} \otimes \rho_N^{\mathcal{W}}\big) \circ \Delta_N^{\mathcal{W},\mathrm{DR}}(z_{m,\zeta_N^a})\\ &= \Big[(X_0^N - 1)^{m-1} X_0^a (1 - X_1) + (Y_0^N - 1)^{m-1} Y_0^a (1 - Y_1)\\ &\qquad + \sum_{\substack{1\leq k\leq m-1\\0\leq b\leq N-1}} (X_0^N - 1)^{k-1} X_0^b (1 - X_1)(Y_0^N - 1)^{m-k-1} Y_0^{a-b}(1 - Y_1)\Big]_m\end{aligned}$$

On the other hand, by taking $P(X_0, X_0^{-1}) = X_0^a$ in (10), we obtain that

$$\begin{aligned}\Delta^{\mathcal{W},\mathrm{B}}\big((X_0^N - 1)^{m-1} X_0^a (1 - X_1)\big) &= (X_0^N - 1)^{m-1} X_0^a (1 - X_1) + (Y_0^N - 1)^{m-1} Y_0^a (1 - Y_1)\\ &+ \widetilde{A}(X_0, Y_0)(1 - X_1)(1 - Y_1) + \widetilde{B}(X_0, Y_0)(1 - X_1)(1 - Y_1) + \widetilde{C}(X_0, Y_0)(1 - X_1)(1 - Y_1),\end{aligned}$$

where

$$\widetilde{A}(X_0, Y_0) := -(X_0^N - 1)^{m-1} X_0^a - (Y_0^N - 1)^{m-1} Y_0^a,$$

$$\widetilde{B}(X_0, Y_0) := (X_0^N - 1)^{m-1} \frac{X_0^{a+1} - Y_0^{a+1}}{X_0 - Y_0},$$

$$\widetilde{C}(X_0, Y_0) := \frac{(X_0^N - 1)^{m-1} - (Y_0^N - 1)^{m-1}}{X_0 - Y_0} Y_0^{a+1}.$$

Thanks to (12), (13) and (14), it follows that

$$\widetilde{A}(X_0, Y_0)(1 - X_1)(1 - Y_1) \in \mathcal{F}^{m+1}(\mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}}),$$

$$\widetilde{B}(X_0, Y_0)(1 - X_1)(1 - Y_1) \in \mathcal{F}^{m+1}(\mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}}),$$

$$\widetilde{C}(X_0, Y_0)(1 - X_1)(1 - Y_1) \in \mathcal{F}^{m}(\mathcal{W}_N^{\mathrm{B}} \otimes \mathcal{W}_N^{\mathrm{B}}).$$

Therefore, thanks to equality (15), we obtain

$$\begin{aligned}&\mathrm{gr}(\Delta^{\mathcal{W},\mathrm{B}})\circ\rho_N^{\mathcal{W}}(z_{m,\zeta_N^a})\\&=\Big[(X_0^N-1)^{m-1}X_0^a(1-X_1)+(Y_0^N-1)^{m-1}Y_0^a(1-Y_1)+\widetilde{C}(X_0,Y_0)(1-X_1)(1-Y_1)\Big]_m.\end{aligned}$$

One checks that

$$\begin{aligned}\widetilde{C}(X_0,Y_0)&=\left(\sum_{k=1}^{m-1}(X_0^N-1)^{k-1}(Y_0^N-1)^{m-k-1}\right)\left(\sum_{b=0}^{N-1}X_0^bY_0^{N-1-b}\right)Y_0^{a+1}\\&=\sum_{\substack{1\le k\le m-1\\0\le b\le N-1}}(X_0^N-1)^{k-1}X_0^b(Y_0^N-1)^{m-k-1}Y_0^{N+a-b}.\end{aligned}$$

Finally,

$$\begin{aligned}&\mathrm{gr}(\Delta^{\mathcal{W},\mathrm{B}})\circ\rho_N^{\mathcal{W}}(z_{m,\zeta_N^a})\\&=\Big[(X_0^N-1)^{m-1}X_0^a(1-X_1)+(Y_0^N-1)^{m-1}Y_0^a(1-Y_1)\\&\qquad+\sum_{\substack{1\le k\le m-1\\0\le b\le N-1}}(X_0^N-1)^{k-1}X_0^b(1-X_1)(Y_0^N-1)^{m-k-1}Y_0^{a-b}(1-Y_1)\Big]_m.\end{aligned}$$

This concludes the proof. □

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References

[DeT] P. Deligne and T. Terasoma. Harmonic shuffle relation for associators. Preprint, 2005.

[Del10] P. Deligne. Le groupe fondamental unipotent motivique de $\mathbf{G}_m-\mu_N$, pour $N=2,3,4,6$ ou 8. *Publ. Math. Inst. Hautes Études Sci.*, 112:101–141, 2010.

[EF1] B. Enriquez and H. Furusho. The Betti side of the double shuffle theory. I. The harmonic coproducts. *Selecta Mathematica*, 27(5):79, 2021.

[EF2] B. Enriquez and H. Furusho. The Betti side of the double shuffle theory. II. Double shuffle relations for associators. *Selecta Mathematica*, 29(3):3, 2023.

[Gon05] A. B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. *Duke Math. J.*, 128(2):209–284, 2005.

[Rac02] G. Racinet. Doubles mélanges des polylogarithmes multiples aux racines de l'unité. *Publications mathématiques de l'IHÉS*, 95(1):185–231, 2002.

[Yad] K. Yaddaden. *Algebraic structures associated to double shuffle relations between multiple polylogarithm values at roots of unity.* PhD thesis, Université de Strasbourg, Strasbourg, France, 2023.

[Yad1] K. Yaddaden. Crossed product interpretation of the double shuffle Lie algebra attached to a finite abelian group. *Publications of the Research Institute for Mathematical Sciences*, 60(3):455–505, 2024.

[Yad2] K. Yaddaden. The cyclotomic double shuffle torsor in terms of betti and de rham coproducts. *Journal of Pure and Applied Algebra*, 228(5):107354, 2024.

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