THE CYCLOTOMIC DOUBLE SHUFFLE TORSOR IN TERMS OF BETTI AND DE RHAM COPRODUCTS

YADDADEN KHALEF

Abstract. In order to describe the double shuffle and regularization relations between multiple polylogarithm values at N^{th} roots of unity, Racinet attached to each finite cyclic group G of order N and each group embedding $\iota : G \to \mathbb{C}^\times$, a $\mathbb{Q}$-scheme DMR^ι which associates to each commutative $\mathbb{Q}$-algebra $\mathbf{k}$, a set $\mathsf{DMR}^\iota(\mathbf{k})$ that can be decomposed as a disjoint union of sets $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ with $\lambda \in \mathbf{k}$. He also exhibited a $\mathbb{Q}$-group scheme DMR^G_0 and showed, for any commutative $\mathbb{Q}$-algebra $\mathbf{k}$ and any $\lambda \in \mathbf{k}^\times$, that $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ is a torsor for the action of $\mathsf{DMR}^G_0(\mathbf{k})$. Then, Enriquez and Furusho showed for $N = 1$ that a subscheme $\mathsf{DMR}^\iota_\times$ of DMR^ι is a torsor of isomorphisms relating "de Rham" and "Betti" objects. In previous work, we reformulated Racinet's construction in terms of crossed products and identified Racinet's coproduct with a coproduct $\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G$ defined on a module $\widehat{\mathcal{M}}^{\mathrm{DR}}_G$ over an algebra $\widehat{\mathcal{W}}^{\mathrm{DR}}_G$, which is equipped with its own coproduct $\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G$. In this paper, we define the main ingredients for a generalization of Enriquez and Furusho's result to any $N \geq 1$: we exhibit a module $\widehat{\mathcal{M}}^{\mathrm{B}}_N$ over an algebra $\widehat{\mathcal{W}}^{\mathrm{B}}_N$ and we prove the existence of two compatible coproducts $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N$ and $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N$ on $\widehat{\mathcal{W}}^{\mathrm{B}}_N$ and $\widehat{\mathcal{M}}^{\mathrm{B}}_N$ respectively such that $\mathsf{DMR}^\iota_\times$ is contained in the torsor of isomorphisms relating $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N$ (resp. $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N$) to $\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G$ (resp. $\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G$).

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Introduction

A *multiple L-value* (MLV in short) is a complex number defined by the following series

$$L_{(k_1,\dots,k_r)}(z_1,\dots,z_r) := \sum_{0<m_1<\cdots<m_r} \frac{z_1^{k_1}\cdots z_r^{k_r}}{m_1^{k_1}\cdots m_r^{k_r}}$$

where $r, k_1, \dots, k_r \in \mathbb{N}^*$ and $z_1, \dots, z_r$ in μ_N the group of N^{th} roots of unity in $\mathbb{C}$, where N is an integer ≥ 1. This series converges if and only if $(k_r, z_r) \neq (1,1)$. These values satisfy a set of algebraic relations; our main interest here are the *double shuffle and regularisation* ones.

Understanding the double shuffle and regularisation relations has been greatly improved thanks to Racinet's work [Rac]. He generalises the group μ_N to a finite cyclic group G and attaches to each pair (G,ι) of a finite cyclic group G and a group injection $\iota : G \to \mathbb{C}^\times$, a $\mathbb{Q}$-scheme DMR^ι which associates to each commutative $\mathbb{Q}$-algebra $\mathbf{k}$, a set $\mathsf{DMR}^\iota(\mathbf{k})$ that can be decomposed as a disjoint union of sets $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ for $\lambda \in \mathbf{k}$ (see [Rac, Definition 3.2.1]). The double shuffle and regularisation relations on MLVs are then encoded in the statement that a suitable generating series of these values belongs to the set $\mathsf{DMR}^{\iota_{can}}_{i2\pi}(\mathbb{C})$ where $\iota_{can} : G = \mu_N \to \mathbb{C}^\star$ is the canonical embedding. Racinet also proved that for any pair (G,ι), the set $\mathsf{DMR}^\iota_0(\mathbf{k})$ equipped with the "twisted Magnus" product (see (1.13)) is a group that is independent of the choice of ι. It is therefore denoted $\mathsf{DMR}^G_0(\mathbf{k})$.

The main result of Racinet in [Rac, Theorem I] is that, for each pair (λ,ι) where $\lambda \in \mathbf{k}^\times$ and $\iota : G \hookrightarrow \mathbb{C}^\times$, the set $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ is equipped with a torsor structure for the action of the group $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$. For any $\iota : G \hookrightarrow \mathbb{C}^\times$, this yields a torsor structure on the set $\mathsf{DMR}^\iota_\times(\mathbf{k}) := \bigsqcup_{\lambda\in\mathbf{k}^\times} \mathsf{DMR}^\iota_\lambda(\mathbf{k})$ for the action of a semidirect product group $\mathbf{k}^\times \ltimes \mathsf{DMR}^G_0(\mathbf{k})$ (see Proposition 1.3.10). This gives rise to a torsor structure on $\mathsf{DMR}_\times(\mathbf{k}) := \bigsqcup_\iota \mathsf{DMR}^\iota_\times(\mathbf{k})$ (where ι runs over all group embeddings from G to $\mathbb{C}^\times$) for the action of the semidirect product group $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{DMR}^G_0(\mathbf{k})$ (see Proposition 1.4.14).

On the other hand, we introduced in [Yad] a *crossed product* formalism and showed that Racinet's objects can be expressed within it. This constitutes the "de Rham" side of the double shuffle theory. In this framework, the crossed product algebra is identified to a topological $\mathbf{k}$-algebra $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ defined by a presentation with generators and relations (see Proposition 1.1.1). Next, Racinet's objects are given in the form of a subalgebra $\widehat{\mathcal{W}}^{\mathrm{DR}}_G$ of $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ and a quotient module $\widehat{\mathcal{M}}^{\mathrm{DR}}_G$ of the left regular $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$-module. The algebra $\widehat{\mathcal{W}}^{\mathrm{DR}}_G$ is equipped with a Hopf algebra coproduct $\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G$ and the module $\widehat{\mathcal{M}}^{\mathrm{DR}}_G$ is equipped with a compatible coalgebra coproduct $\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G$.

Following the stabilizer interpretation of $\mathsf{DMR}^G_0(\mathbf{k})$ given by Enriquez and Furusho in [EF0], we defined two stabilizers $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k})$ and $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k})$ for the action of grouplike elements equipped with the twisted Magnus product and therefore obtained the following chain of inclusions

$$\mathsf{DMR}^G_0(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k}),$$

which is a generalisation of the $G = \{1\}$ result of [EF2, Theorem 3.1]. This enables us to obtain the following semidirect product group chain of inclusions

Result I (Corollary 2.2.5)**.**

$$\begin{array}{ccc}(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}_0^G(\mathbf{k})\subset & (\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k}) \\ & \cap \\ & (\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})\end{array}$$

For $G = \{1\}$ Enriquez and Furusho introduced a "Betti" formalism of the double shuffle theory in [EF1]. It is based on the filtered algebra $\mathcal{V}^{\mathrm{B}}$, which denotes the group algebra over $\mathbf{k}$ of the free group of rank 2 denoted F_2 with generators X_0 and X_1 and equipped with the filtration induced by the augmentation ideal. The completion $\widehat{\mathcal{V}}^{\mathrm{B}}$ is a topological $\mathbf{k}$-algebra. Next, we have a Hopf algebra $(\widehat{\mathcal{W}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}})$ which consists of a subalgebra $\widehat{\mathcal{W}}^{\mathrm{B}}$ of $\widehat{\mathcal{V}}^{\mathrm{B}}$ linearly generated by $1 \in \widehat{\mathcal{V}}^{\mathrm{B}}$ and the left ideal generated by X_1-1. It is presented as an algebra with generators X_1, X_1^{-1}, $Y_n^+ = -(X_0-1)^{n-1}X_0(X_1-1)$ and $Y_n^- = -(X_0^{-1}-1)^{n-1}X_0^{-1}(X_1^{-1}-1)$ for $n \in \mathbb{N}^*$, with relation $X_1X_1^{-1} = X_{-1}X_1 = 1$. In addition, we have a Hopf algebra coproduct $\widehat{\Delta}^{\mathcal{W},\mathrm{B}} : \widehat{\mathcal{W}}^{\mathrm{B}} \to (\widehat{\mathcal{W}}^{\mathrm{B}})^{\hat{\otimes}2}$ given by

$$\widehat{\Delta}^{\mathcal{W},\mathrm{B}}(X_1^{\pm 1}) = X_1^{\pm 1}\otimes X_1^{\pm 1} \text{ and for } n\in\mathbb{N}^*, \widehat{\Delta}^{\mathcal{W},\mathrm{B}}(Y_n^{\pm}) = Y_n^{\pm}\otimes 1 + 1\otimes Y_n^{\pm} + \sum_{\substack{k,l\in\mathbb{N}^*\\ k+l=n}} Y_k^{\pm}\otimes Y_l^{\pm}.$$

Finally, we have a coalgebra $(\widehat{\mathcal{M}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}})$ which consists of a quotient module $\widehat{\mathcal{M}}^{\mathrm{B}} = \widehat{\mathcal{V}}^{\mathrm{B}}/\widehat{\mathcal{V}}^{\mathrm{B}}(X_0-1)$ isomorphic to $\widehat{\mathcal{W}}^{\mathrm{B}}$, as a $\mathbf{k}$-module (see [EF1, (2.1.1)]) together with a coalgebra coproduct $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}$ compatible with the coproduct $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}$.

In §3, we construct analogues of the Hopf algebra $(\widehat{\mathcal{W}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}})$ and of the module-coalgebra $(\widehat{\mathcal{M}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}})$, for a finite cyclic group G of order N. It is based on the filtered algebra $\mathcal{V}_N^{\mathrm{B}}$, which denotes the group algebra $\mathbf{k}F_2$ equipped with the filtration induced by the ideal $\ker(\mathbf{k}F_2 \to \mathbf{k}\mu_N)$; where $\mathbf{k}F_2 \to \mathbf{k}\mu_N$ is the algebra morphism induced by the group morphism $F_2 \to \mu_N$ given by $X_0 \mapsto e^{\frac{i2\pi}{N}}$ and $X_1 \mapsto 1$. Its completion is the inverse limit of the projective system induced by the filtration and is denoted $\widehat{\mathcal{V}}_N^{\mathrm{B}}$. It is a topological algebra isomorphic to $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ (see Proposition-Definition 3.1.8). Next, we define a filtered algebra $\mathcal{W}_N^{\mathrm{B}}$ which is linearly generated by $1 \in \mathcal{V}_N^{\mathrm{B}}$ and the left ideal generated by $X_1 - 1$ and whose filtration is induced by that of $\mathcal{V}_N^{\mathrm{B}}$. Its completion $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ is isomorphic to $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ (see Proposition-Definition 3.1.15). We also define a filtered module $\mathcal{M}_N^{\mathrm{B}}$ which consists of the quotient module $\mathbf{k}F_2/\mathbf{k}F_2(X_0-1)$ and whose filtration is induced by that of $\mathcal{V}_N^{\mathrm{B}}$. Its completion $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ is isomorphic to $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ (see Proposition-Definition 3.1.25). We then have compatible Hopf algebra and coalgebra structures on $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ respectively thanks to the following result:

Result II (Theorem 3.2.4 and Corollary 3.2.6)**.** *There exists a topological $\mathbf{k}$-algebra morphism $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to (\widehat{\mathcal{W}}_N^{\mathrm{B}})^{\hat{\otimes}2}$ and a topological $\mathbf{k}$-module morphism $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to (\widehat{\mathcal{M}}_N^{\mathrm{B}})^{\hat{\otimes}2}$ that endows $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ respectively with compatible Hopf algebra and coalgebra structures.*

Finally, in §4, we deduce the following result:

Result III (Theorem 4.2.1)**.** $\mathsf{DMR}_\times$ *is contained in the torsor of isomorphisms relating* $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N$ *(resp.* $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N$*) to* $\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G$ *(resp.* $\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G$*).*

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Notation. Throughout this paper, G is a (multiplicative) finite cyclic group of order N and $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra. For a $\mathbf{k}$-algebra A, an element $x \in A$ and a left A-module M we consider:

- $\ell_x : M \to M$ to be the $\mathbf{k}$-module endomorphism defined by $m \mapsto xm$ and if x is invertible, then ℓ_x is an automorphism.
- $r_x : A \to A$ to be the $\mathbf{k}$-module endomorphism defined by $a \mapsto ax$ and if x is invertible, then r_x is an automorphism.
- $\mathrm{Ad}_x : A \to A$ to be the $\mathbf{k}$-algebra automorphism defined by $a \mapsto xax^{-1}$ with $x \in A^\times$.

0. Some categories of algebra-modules

First, let us recall various categories introduced in [EF4] that will be used throughout this paper:

- $\mathbf{k}$-mod is the category of $\mathbf{k}$-modules.
- $\mathbf{k}$-alg is the category of $\mathbf{k}$-algebras.
- $\mathbf{k}$-alg-mod is the category of pairs (A, M) where A is a $\mathbf{k}$-algebra and M is a left A-module.
- $\mathbf{k}$-coalg is the category of coassociative cocommutative coalgebras over $\mathbf{k}$.
- $\mathbf{k}$-Hopf is the category of Hopf algebras over $\mathbf{k}$.
- $\mathbf{k}$-HAMC is the category of Hopf-Algebra-Module-Coalgebras where objects are pairs $\big((A, \Delta^A), (M, \Delta^M)\big)$ where (A, Δ^A) is a Hopf algebra and (M, Δ^M) is a coalgebra such that
  - ▶ The pair (A, M) is an algebra-module.
  - ▶ For $(a, m) \in A \times M$, we have $\Delta^M(am) = \Delta^A(a)\Delta^M(m)$.
- $\mathbf{k}$-$\mathrm{mod}_{\mathrm{top}}$ is the category of topological $\mathbf{k}$-modules with objects pairs $(M, (\mathcal{F}^i M)_{i\in\mathbb{N}})$, where M is a $\mathbf{k}$-module and $(\mathcal{F}^i M)_{i\in\mathbb{N}}$ is a decreasing filtration of M such that the map $M \to \varprojlim M/F^iM$ is a $\mathbf{k}$-module isomorphism, i.e. M is complete and separated for the topology defined by the filtration $(\mathcal{F}^i M)_{i\in\mathbb{N}}$. It is equipped with a tensor product $\hat{\otimes}$, with respect to which it is a symmetric tensor category.
- $\mathbf{k}$-$\mathrm{alg}_{\mathrm{top}}$ is the category of topological $\mathbf{k}$-algebras. i. e. algebras in the category $\mathbf{k}$-$\mathrm{mod}_{\mathrm{top}}$ in the sense of [McL].
- $\mathbf{k}$-alg-$\mathrm{mod}_{\mathrm{top}}$ is the category of topological $\mathbf{k}$-algebra-modules. i. e. $\mathbf{k}$-algebra-modules in the category $\mathbf{k}$-$\mathrm{mod}_{\mathrm{top}}$ in the sens of [McL].
- $\mathbf{k}$-$\mathrm{coalg}_{\mathrm{top}}$ is the category of topological $\mathbf{k}$-coalgebras. i. e. coalgebras in the category $\mathbf{k}$-$\mathrm{mod}_{\mathrm{top}}$ in the sens of [McL].
- $\mathbf{k}$-$\mathrm{Hopf}_{\mathrm{top}}$ is the category of topological $\mathbf{k}$-Hopf algebras. i. e. Hopf algebras in the category $\mathbf{k}$-$\mathrm{mod}_{\mathrm{top}}$ in the sens of [McL].
- $\mathbf{k}$-$\mathrm{HAMC}_{\mathrm{top}}$ is the category of topological Hopf-Algebra-Module-Coalgebras. i. e. Hopf-Algebra-Module-Coalgebras in the category $\mathbf{k}$-$\mathrm{mod}_{\mathrm{top}}$ in the sens of [McL].

Finally, let $\mathcal{C}$ be a symmetric tensor category and O an object of $\mathcal{C}$. We define $\mathrm{Cop}_{\mathcal{C}}(O)$ to be the set of morphisms $D : O \to O^{\hat{\otimes}2}$. One checks that the group $\mathrm{Aut}_{\mathcal{C}}(O)$ acts on $\mathrm{Cop}_{\mathcal{C}}(O)$ by

$$\alpha \cdot D := \alpha^{\otimes 2} \circ D \circ \alpha^{-1}, \tag{0.1}$$

with $\alpha \in \mathrm{Aut}_{\mathcal{C}}(O)$ and $D \in \mathrm{Cop}_{\mathcal{C}}(O)$.

1. The double shuffle torsors

In this section, we recall the various double shuffle torsors arising from Racinet's work in [Rac]. In §1.1, we recall the basic framework of Racinet's formalism. Namely, two Hopf algebras $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$ and $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$, a coalgebra $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$ and a group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$. Additionally, we also recall the basic material of the crossed product formalism introduced in [Yad] which consists of an algebra $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ and its relation with a Hopf algebra $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$ isomorphic to the Hopf algebra $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$ and a coalgebra $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$ isomorphic to the coalgebra $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$. In §1.2, we introduce the double shuffle set $\mathsf{DMR}_\lambda^\iota(\mathbf{k})$ for $\lambda \in \mathbf{k}$ and $\iota : G \to \mathbb{C}^\times$ an injective group morphism, which is a torsor over the double shuffle group $\mathsf{DMR}_0^G(\mathbf{k})$, a subgroup of $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ ([Rac]). In §1.3, we define a set $\mathsf{DMR}_\times^\iota(\mathbf{k}) = \bigsqcup_{\lambda\in\mathbf{k}^\times} \mathsf{DMR}_\lambda^\iota(\mathbf{k})$ and show that it is a torsor for a group given by a semidirect product arising from an action of $\mathbf{k}^\times$. Finally, in §1.4, we define a set $\mathsf{DMR}_\times(\mathbf{k}) = \bigsqcup_\iota \mathsf{DMR}_\times^\iota(\mathbf{k})$ where ι runs over all injections $G \to \mathbb{C}^\times$ and show that it is a torsor for a group given by semidirect product arising from an action of $\mathrm{Aut}(G) \times \mathbf{k}^\times$.

1.1. **Preliminaries.**

1.1.1. *Basic objects of Racinet's formalism.* Let $\mathbf{k}\langle\langle X\rangle\rangle$ be the free noncommutative associative series algebra with unit over the alphabet $X := \{x_0\} \sqcup \{x_g | g \in G\}$. It is complete graded with $\deg(x_0) = \deg(x_g) = 1$ for $g \in G$. This algebra provides an object in $\mathbf{k}$-$\mathrm{Hopf}_{\mathrm{top}}$ when equipped with the coproduct $\widehat{\Delta} : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle^{\hat{\otimes}2}$, which is the unique topological $\mathbf{k}$-algebra morphism given by $\widehat{\Delta}(x_g) = x_g \otimes 1 + 1 \otimes x_g$, for any $g \in G \sqcup \{0\}$ ([Rac, §2.2.3]). Let then $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ be the set of grouplike elements of $\mathbf{k}\langle\langle X\rangle\rangle$ for the coproduct $\widehat{\Delta}$, i.e. the set

$$\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) = \{\Psi \in \mathbf{k}\langle\langle X\rangle\rangle^\times \,|\, \widehat{\Delta}(\Psi) = \Psi \otimes \Psi\}, \tag{1.1}$$

where $\mathbf{k}\langle\langle X\rangle\rangle^\times$ denotes the set of invertible elements of $\mathbf{k}\langle\langle X\rangle\rangle$. Since $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$ is a Hopf algebra, $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ is a group for the product of $\mathbf{k}\langle\langle X\rangle\rangle$.

The group G acts on $\mathbf{k}\langle\langle X\rangle\rangle$ by topological $\mathbf{k}$-algebra automorphisms by $g \mapsto t_g$, where for any $g \in G$, the topological $\mathbf{k}$-algebra automorphism t_g is given by $t_g(x_0) = x_0$, $t_g(x_h) = x_{gh}$ for $h \in G$ ([Rac, §3.1.1]). For any $g \in G$, we have

$$\widehat{\Delta} \circ t_g = t_g^{\otimes 2} \circ \widehat{\Delta}, \tag{1.2}$$

this can be verified by checking this identity on generators since both sides are given as a composition of $\mathbf{k}$-algebra morphisms. As a consequence, for any $g \in G$, the $\mathbf{k}$-algebra automorphism $t_g : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$ restricts to a group automorphism of $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Let $\mathbf{q}$ be the $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ given on the topological $\mathbf{k}$-module basis $\big(x_0^{n_1}x_{g_1}x_0^{n_2}x_{g_2}\cdots x_0^{n_r}x_{g_r}x_0^{n_{r+1}}\big)_{\substack{r,n_1,\dots,n_{r+1}\in\mathbb{N}\\ g_1,\dots,g_r\in G}}$ of $\mathbf{k}\langle\langle X\rangle\rangle$ ([Rac, §2.2.7]) by

$$\mathbf{q}(x_0^{n_1}x_{g_1}x_0^{n_2}x_{g_2}\cdots x_0^{n_r}x_{g_r}x_0^{n_{r+1}}) = x_0^{n_1}x_{g_1}x_0^{n_2}x_{g_2g_1^{-1}}\cdots x_0^{n_r}x_{g_rg_{r-1}^{-1}}x_0^{n_{r+1}}.$$

For $(n,g)\in\mathbb{N}_{>0}\times G$, set $y_{n,g}:=x_0^{n-1}x_g$. Let $Y:=\{y_{n,g}|(n,g)\in\mathbb{N}_{>0}\times G\}$. We define $\mathbf{k}\langle\langle Y\rangle\rangle$ to be the topologically free $\mathbf{k}$-algebra over Y, where for every $(n,g)\in\mathbb{N}_{>0}\times G$, the element $y_{n,g}$ is of degree n. One shows that $\mathbf{k}\langle\langle Y\rangle\rangle$ is equal to the $\mathbf{k}$-subalgebra $\mathbf{k}\oplus\bigoplus_{g\in G}\mathbf{k}\langle\langle X\rangle\rangle x_g$ of $\mathbf{k}\langle\langle X\rangle\rangle$ ([Rac, §2.2.5] and [EF0, §2.2]).

Let $\widehat{\Delta}_\star^{\mathrm{alg}}:\mathbf{k}\langle\langle Y\rangle\rangle\to(\mathbf{k}\langle\langle Y\rangle\rangle)^{\hat{\otimes}2}$ be the unique topological $\mathbf{k}$-algebra morphism such that for any $(n,g)\in\mathbb{N}_{>0}\times G$

$$\widehat{\Delta}_\star^{\mathrm{alg}}(y_{n,g}) = y_{n,g}\otimes 1 + 1\otimes y_{n,g} + \sum_{\substack{k=1\\ h\in G}}^{n-1} y_{k,h}\otimes y_{n-k,gh^{-1}}. \tag{1.3}$$

The element $\widehat{\Delta}_\star^{\mathrm{alg}}\in\mathrm{Cop}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\mathbf{k}\langle\langle Y\rangle\rangle)$ is called the *harmonic coproduct* ([Rac, §2.3.1]) and the pair $(\mathbf{k}\langle\langle Y\rangle\rangle,\widehat{\Delta}_\star^{\mathrm{alg}})$ is an object of $\mathbf{k}$-$\mathrm{Hopf}_{\mathrm{top}}$.

Let us consider the topological $\mathbf{k}$-module quotient $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$. The pair $(\mathbf{k}\langle\langle Y\rangle\rangle,\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)$ is an object of the category $\mathbf{k}$-alg-$\mathrm{mod}_{\mathrm{top}}$. The restriction to $\mathbf{k}\langle\langle Y\rangle\rangle$ of the projection morphism $\pi_Y:\mathbf{k}\langle\langle X\rangle\rangle\to\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ is an isomorphism, therefore $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ is free of rank 1 over $\mathbf{k}\langle\langle Y\rangle\rangle$. It follows that there is a topological $\mathbf{k}$-module morphism $\widehat{\Delta}_\star^{\mathrm{mod}}\in\mathrm{Cop}_{\mathbf{k}\text{-mod}_{\mathrm{top}}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)$ uniquely defined by the condition that the diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\widehat{\Delta}_\star^{\mathrm{alg}}} & \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2} \\ \pi_Y\downarrow & & \downarrow\pi_Y^{\otimes 2} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\widehat{\Delta}_\star^{\mathrm{mod}}} & (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes}2} \end{array} \tag{1.4}$$

commutes. The pair $(\widehat{\Delta}_\star^{\mathrm{alg}},\widehat{\Delta}_\star^{\mathrm{mod}})$ is an element of $\mathrm{Cop}_{\mathbf{k}\text{-alg-mod}_{\mathrm{top}}}(\mathbf{k}\langle\langle Y\rangle\rangle,\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)$. The pair $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0,\widehat{\Delta}_\star^{\mathrm{mod}})$ is an object of $\mathbf{k}$-$\mathrm{coalg}_{\mathrm{top}}$ and the pair $\big((\mathbf{k}\langle\langle Y\rangle\rangle,\widehat{\Delta}_\star^{\mathrm{alg}}),(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0,\widehat{\Delta}_\star^{\mathrm{mod}})\big)$ is an object of $\mathbf{k}$-$\mathrm{HAMC}_{\mathrm{top}}$.

1.1.2. *Basic objects of the crossed product formalism.* Let $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ be the complete graded algebra generated by $\{e_0,e_1\}\sqcup G$ where e_0 and e_1 are of degree 1 and elements $g\in G$ are of degree 0 satisfying the relations:

(i) $g\times h=gh$; (ii) $1=1_G$; (iii) $g\times e_0=e_0\times g$;

for any $g,h\in G$; where "$\times$" is the algebra multiplication[1]([Yad, §2.1.1]).

Recall that the map $g\mapsto t_g$ defines an action of G on $\mathbf{k}\langle\langle X\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms. One then considers the crossed product algebra $\mathbf{k}\langle\langle X\rangle\rangle\rtimes G$ for this

[1]which we will no longer denote if there is no risk of ambiguity.

action, which is the **k**-module $\mathbf{k}\langle\langle X\rangle\rangle \otimes \mathbf{k}G$ equipped with the product given for any $a, b \in \mathbf{k}\langle\langle X\rangle\rangle$ and any $g, h \in G$ by ([Bou07, Chapter 3, Page 180, Exercise 11])

$$(a \otimes g) * (b \otimes h) = a\, t_g(b) \otimes gh. \tag{1.5}$$

Proposition 1.1.1 ([Yad, Proposition 2.1.3])**.** *There is a unique* **k***-algebra isomorphism* $\beta : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ *such that* $x_0 \otimes 1 \mapsto e_0$ *and for* $g \in G$*,* $x_g \otimes 1 \mapsto -ge_1g^{-1}$ *and* $1 \otimes g \mapsto g$*.*

Let $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ be the complete graded **k**-subalgebra of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ given by

$$\widehat{\mathcal{W}}_G^{\mathrm{DR}} := \mathbf{k} \oplus \widehat{\mathcal{V}}_G^{\mathrm{DR}} e_1. \tag{1.6}$$

It is freely generated by the family

$$Z = \{z_{n,g} := -e_0^{n-1} g e_1 \,|\, (n, g) \in \mathbb{N}_{>0} \times G\},$$

with $\deg(z_{n,g}) = n$ ([Yad, Proposition 2.1.5.(b)]). As a consequence, there is a unique **k**-algebra isomorphism $\varpi : \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ given for $(n, g) \in \mathbb{N}_{>0} \times G$ by $y_{n,g} \mapsto z_{n,g}$.

One then has a unique topological **k**-algebra morphism $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} : \widehat{\mathcal{W}}_G^{\mathrm{DR}} \to (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes} 2}$ such that for any $(n, g) \in \mathbb{N}_{>0} \times G$

$$\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,g}) = z_{n,g} \otimes 1 + 1 \otimes z_{n,g} + \sum_{\substack{k=1\\ h\in G}}^{n-1} z_{k,h} \otimes z_{n-k,gh^{-1}}. \tag{1.7}$$

The coproduct $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ is an element of $\mathrm{Cop}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})$. The pair $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$ is an object in the category **k**-$\mathrm{Hopf}_{\mathrm{top}}$ and the **k**-algebra isomorphism $\varpi : \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is an isomorphism between the Hopf algebras $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$ and $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$.

Let $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ be the complete graded **k**-module given by

$$\widehat{\mathcal{M}}_G^{\mathrm{DR}} := \widehat{\mathcal{V}}_G^{\mathrm{DR}} \Big/ \Big(\widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g-1)\Big).$$

Let 1_{DR} be the class of $1 \in \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ in $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$. The map $- \cdot 1_{\mathrm{DR}} : \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a surjective **k**-module morphism with kernel $\widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g-1)$. In addition, the pair $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ is an object in the category **k**-alg-$\mathrm{mod}_{\mathrm{top}}$. Moreover, one deduces from [Yad, Proposition 2.1.6] that there is a unique **k**-module isomorphism $\kappa : \mathbf{k}\langle\langle X\rangle\rangle / \mathbf{k}\langle\langle X\rangle\rangle x_0 \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ determined by the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle \pi_Y \circ \mathbf{q}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\ \mathbf{k}\langle\langle X\rangle\rangle / \mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\kappa} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{1.8}$$

On the other hand, the following diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\quad\varpi\quad} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\quad\kappa\quad} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{1.9}$$

commutes ([Yad, Corollary 2.1.8]). As a consequence, the map $-\cdot 1_{\mathrm{DR}} : \widehat{\mathcal{W}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a $\mathbf{k}$-module isomorphism since all other arrows of Diagram (1.9) are isomorphisms. In addition, we obtain the following result

Lemma 1.1.2. *The pair* $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ *is an object of the category* $\mathbf{k}$-alg-mod$_{\mathrm{top}}$. *Moreover,* $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ *is free* $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$*-module of rank* 1.

Proof. The first statement follows from the fact that $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ is the pull-back of the $\mathbf{k}$-algebra-module $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ by the $\mathbf{k}$-algebra morphism $\widehat{\mathcal{W}}_G^{\mathrm{DR}} \hookrightarrow \widehat{\mathcal{V}}_G^{\mathrm{DR}}$.
The second statement comes from the fact that $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ is a free $\mathbf{k}\langle\langle Y\rangle\rangle$-module of rank 1 thanks to the commutativity of Diagram (1.9). ☐

This enables us to construct a topological $\mathbf{k}$-module morphism $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \in \mathrm{Cop}_{\mathbf{k}\text{-mod}_{\mathrm{top}}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}})$ uniquely defined such that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\quad\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\quad} & (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes}2} \\ {\scriptstyle -\cdot 1_{\mathrm{DR}}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}^{\otimes 2}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{\quad\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\quad} & (\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\hat{\otimes}2} \end{array} \tag{1.10}$$

commutes, thanks to Lemma 1.1.2 and the free rank 1 property of the $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$-module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$. The pair $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$ is an object in the category $\mathbf{k}$-coalg$_{\mathrm{top}}$ and the $\mathbf{k}$-module isomorphism $\kappa : \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is an isomorphism of coalgebras $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$ and $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$.

Lemma 1.1.3. *The pair* $\left(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)$ *is an element of* $\mathrm{Cop}_{\mathbf{k}\text{-alg-mod}_{\mathrm{top}}}\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}\right)$.

Proof. Let $w \in \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ and $m \in \widehat{\mathcal{M}}_G^{\mathrm{DR}}$. Thanks to Lemma 1.1.2 there is a unique $w' \in \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ such that $m = w' \cdot 1_{\mathrm{DR}}$. We have

$$\begin{aligned} \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}(w \cdot m) &= \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}(ww' \cdot 1_{\mathrm{DR}}) = \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(ww') \cdot 1_{\mathrm{DR}}^{\otimes 2} = \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(w)\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(w') \cdot 1_{\mathrm{DR}}^{\otimes 2} \\ &= \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(w)\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}(w' \cdot 1_{\mathrm{DR}}) = \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(w)\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}(m), \end{aligned}$$

where the second and fourth equalities come from the commutativity of Diagram (1.10). ☐

As a consequence, the pair $\left((\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}), (\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})\right)$ is an object of $\mathbf{k}$-HAMC$_{\mathrm{top}}$. In addition, the pair

$$(\varpi, \kappa) : \left((\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}}), (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})\right) \to \left((\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}), (\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})\right)$$

is a morphism of objects of $\mathbf{k}$-HAMC$_{\mathrm{top}}$.

1.2. **The torsor $\mathsf{DMR}^{\iota}_{\lambda}(\mathbf{k})$.** Let us denote $\mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}^{\{\text{words in } x_0,(x_g)_{g\in G}\}}, v \mapsto \big((v|w)\big)_w$ the map such that $v = \sum_w (v|w)w$ (the empty word is equal to 1).

Let $\Gamma : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}[[x]]^{\times}, \Psi \mapsto \Gamma_{\Psi}$ the function[2] given by ([Rac, (3.2.1.2)])

$$\Gamma_{\Psi}(x) := \exp\left(\sum_{n\geq 2} \frac{(-1)^{n-1}}{n}(\Psi|x_0^{n-1}x_1)x^n\right). \tag{1.11}$$

Definition 1.2.1 ([Rac, Definition 3.2.1])**.** Let $\lambda \in \mathbf{k}$ and $\iota : G \to \mathbb{C}^{\times}$ be a group embedding. We define $\mathsf{DMR}^{\iota}_{\lambda}(\mathbf{k})$ to be the set of $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ such that

(i) $(\Psi|x_0) = (\Psi|x_1) = 0$;

(ii) $\widehat{\Delta}^{\mathrm{mod}}_{\star}(\Psi_{\star}) = \Psi_{\star} \otimes \Psi_{\star}$;

(iii) If $|G| \in \{1,2\}, (\Psi|x_0x_1) = -\frac{\lambda^2}{24}$;

(iv) If $|G| \geq 3$, $\left(\Psi|x_{g_\iota} - x_{g_\iota^{-1}}\right) = \frac{|G|-2}{2}\lambda$;

(v) For $k \in \{1,\dots,|G|/2\}$, $\left(\Psi|x_{g_\iota^k} - x_{g_\iota^{-k}}\right) = \frac{|G|-2k}{|G|-2}\left(\Psi|x_{g_\iota} - x_{g_\iota^{-1}}\right)$,

where $g_\iota := \iota^{-1}(e^{\frac{i2\pi}{|G|}})$ and $\Psi_{\star} := \pi_Y \circ \mathbf{q}\left(\Gamma_{\Psi}^{-1}(x_1)\Psi\right) \in \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

Remark 1.2.2.

(i) Thanks to [Rac, §3.2.3], $\mathsf{DMR}^{\iota}_{\lambda}(\mathbf{k})$ is a non-empty set.

(ii) If $|G| \in \{1,2\}$, the embedding ι is unique.

Proposition-Definition 1.2.3 ([Rac, Remark 3.2.2])**.** *For $\lambda = 0$, the condition[3] (iv) of Definition 1.2.1 does not depend of the choice of ι. The set $\mathsf{DMR}^{\iota}_0(\mathbf{k})$ is then denoted $\mathsf{DMR}^{G}_0(\mathbf{k})$ instead.*

Proposition 1.2.4. *Condition (ii) of Definition 1.2.1 is equivalent to*

$$\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G}(\Psi^{\star}) = \Psi^{\star} \otimes \Psi^{\star}, \tag{1.12}$$

where $\Psi^{\star} := \left(\Gamma_{\Psi}^{-1}(-e_1)\beta(\Psi \otimes 1)\right) \cdot 1_{\mathrm{DR}} \in \widehat{\mathcal{M}}^{\mathrm{DR}}_{G}$.

Proof. Thanks to Diagram (1.8), it follows that $\kappa(\Psi_{\star}) = \Psi^{\star}$. Equality (1.12) then follows from the fact that $\kappa : (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}^{\mathrm{mod}}_{\star}) \to (\widehat{\mathcal{M}}^{\mathrm{DR}}_{G}, \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G})$ is a coalgebra isomorphism. □

Recall the set $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ of grouplike elements of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$ given in (1.1). In addition to its usual group structure, it is also a group for the "twisted Magnus" product denoted $\circledast$ and given for any $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ by

$$\Psi \circledast \Phi := \Psi \mathrm{aut}_{\Psi}(\Phi), \tag{1.13}$$

where aut_{Ψ} is the topological $\mathbf{k}$-algebra automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ given by ([EF0], §4.1.3 based on [Rac], §3.1.2)

$$x_0 \mapsto x_0 \qquad \text{and for } g \in G, x_g \mapsto \mathrm{Ad}_{t_g(\Psi^{-1})}(x_g). \tag{1.14}$$

Proposition 1.2.5 ([Rac, Theorem I])**.** *Let $\lambda \in \mathbf{k}$ and $\iota : G \to \mathbb{C}^{\times}$ be a group embedding.*

(i) The pair $(\mathsf{DMR}^{G}_0(\mathbf{k}), \circledast)$ is a subgroup of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$.

[2]This function is related to the classical gamma function as established in [Fu11], page 344 thanks to [Dri90].

[3]This also holds for condition (v).

(ii) The group $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$ *acts freely and transitively on* $\mathsf{DMR}_\lambda^\iota(\mathbf{k})$ *by left multiplication* $\circledast$*.*

1.3. The torsor $\mathsf{DMR}_\times^\iota(\mathbf{k})$.

1.3.1. *Action of the group* $\mathbf{k}^\times$ *on* $\mathbf{k}\langle\langle X\rangle\rangle$*.* The group $\mathbf{k}^\times$ acts on $\mathbf{k}\langle\langle X\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms by

$$\mathbf{k}^\times \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle X\rangle\rangle); \quad \lambda \longmapsto \lambda \bullet - : x_g \mapsto \lambda x_g, \text{ for } g \in G \sqcup \{0\}. \tag{1.15}$$

One checks that, for any $\lambda \in \mathbf{k}^\times$, the automorphism $\lambda \bullet -$ is a Hopf algebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$. In addition, for any $\lambda \in \mathbf{k}^\times$ and any $g \in G$, we have

$$(\lambda \bullet -) \circ t_g = t_g \circ (\lambda \bullet -), \tag{1.16}$$

this can be verified by checking this identity on generators since both sides are given as a composition of $\mathbf{k}$-algebra morphisms.

Proposition 1.3.1. *For any* $\lambda \in \mathbf{k}^\times$*, the map* $\lambda \bullet - : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$ *restricts to a group automorphism of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$*.*

In order the prove this, we will need the following Lemma:

Lemma 1.3.2. *For any* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, we have*

$$(\lambda \bullet -) \circ \mathrm{aut}_\Psi = \mathrm{aut}_{\lambda\bullet\Psi} \circ (\lambda \bullet -). \tag{1.17}$$

Proof. Let $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Since all the morphisms are algebra automorphisms of $\mathbf{k}\langle\langle X\rangle\rangle$, it is enough to check this identity on generators. We have

$$(\lambda \bullet -) \circ \mathrm{aut}_\Psi(x_0) = \lambda \bullet x_0 = \lambda x_0 = \lambda \mathrm{aut}_{\lambda\bullet\Psi}(x_0) = \mathrm{aut}_{\lambda\bullet\Psi}(\lambda x_0) = \mathrm{aut}_{\lambda\bullet\Psi}(\lambda \bullet x_0)$$

and for $g \in G$,

$$\begin{aligned}(\lambda \bullet -) \circ \mathrm{aut}_\Psi(x_g) &= (\lambda \bullet -) \circ \mathrm{Ad}_{t_g(\Psi^{-1})}(x_g) = \mathrm{Ad}_{\lambda\bullet t_g(\Psi^{-1})}(\lambda \bullet x_g) \\ &= \mathrm{Ad}_{t_g(\lambda\bullet\Psi^{-1})}(\lambda \bullet x_g) = \mathrm{aut}_{\lambda\bullet\Psi}(\lambda \bullet x_g),\end{aligned}$$

where the third equality comes from Identity (1.16). □

Proof of Proposition 1.3.1. Let $\lambda \in \mathbf{k}^\times$. Since $\lambda \bullet -$ is a Hopf algebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$, it restricts to a map $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Let $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have

$$\begin{aligned}\lambda \bullet (\Psi \circledast \Phi) &= \lambda \bullet (\Psi \mathrm{aut}_\Psi(\Phi)) = (\lambda \bullet \Psi)\,(\lambda \bullet \mathrm{aut}_\Psi(\Phi)) \\ &= (\lambda \bullet \Psi)\,\mathrm{aut}_{\lambda\bullet\Psi}(\lambda \bullet \Phi) = (\lambda \bullet \Psi) \circledast (\lambda \bullet \Phi),\end{aligned}$$

where the third equality comes from Lemma 1.3.2. This proves that $\lambda \bullet -$ is a group endomorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$. Finally, one has that $(\lambda \bullet -)^{-1} = \lambda^{-1} \bullet -$ and the above computations shows that $(\lambda \bullet -)^{-1}$ is an endomorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ thus proving the statement. □

Proposition 1.3.1 enables us to define the following:

Definition 1.3.3. We denote $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ the semi-direct product of $\mathbf{k}^\times$ and $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ with respect to the action given in Proposition 1.3.1. It consists of the set $\mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ endowed with a group law which will also be denoted $\circledast$ and we have for $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$,

$$(\lambda, \Psi) \circledast (\nu, \Phi) := (\lambda\nu, \Psi \circledast (\lambda \bullet \Phi)). \tag{1.18}$$

1.3.2. *Action of the group $\mathbf{k}^\times$ on crossed product algebras and module.* The group $\mathbf{k}^\times$ acts on $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by $\mathbf{k}$-algebra automorphisms by

$$\mathbf{k}^\times \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}});\ \lambda \longmapsto \lambda \bullet_{\mathcal{V}} - : e_i \mapsto \lambda e_i;\ g \mapsto g,\ \text{ for } i \in \{0,1\} \text{ and } g \in G. \tag{1.19}$$

Lemma 1.3.4. *Let $\lambda \in \mathbf{k}^\times$.*

(i) The following diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\lambda\bullet-} & \mathbf{k}\langle\langle X\rangle\rangle \\ {\scriptstyle \beta\circ(-\otimes 1)}\downarrow & & \downarrow{\scriptstyle \beta\circ(-\otimes 1)} \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{\lambda\bullet_{\mathcal{V}}-} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{1.20}$$

commutes.

(ii) For $\Psi \in \mathbf{k}\langle\langle X\rangle\rangle$, we have

$$\Gamma_{\lambda\bullet\Psi}(-e_1) = \lambda \bullet_{\mathcal{V}} \Gamma_\Psi(-e_1). \tag{1.21}$$

Proof.

(i) Since all arrows are $\mathbf{k}$-algebra morphisms, one easily checks the commutativity of generators.

(ii) It follows from the fact that, for $n \in \mathbb{N}_{>0}$, we have $(\lambda\bullet\Psi|x_0^{n-1}x_1) = \lambda^n(\Psi|x_0^{n-1}x_1)$.

□

Lemma 1.3.5. *Let $\lambda \in \mathbf{k}^\times$.*

(i) The $\mathbf{k}$-algebra automorphism $\lambda \bullet_{\mathcal{V}} -$ of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ restricts to a Hopf algebra automorphism $\lambda \bullet_{\mathcal{W}} -$ of $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$.

(ii) The $\mathbf{k}$-algebra automorphism $\lambda \bullet_{\mathcal{V}} -$ of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ induces a coalgebra automorphism $\lambda \bullet_{\mathcal{M}} -$ of $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$.

Proof.

(i) For $(n, g) \in \mathbb{N}_{>0} \times G$ we have

$$\lambda \bullet_{\mathcal{V}} z_{n,g} = \lambda \bullet_{\mathcal{V}} (-e_0^{n-1}ge_1) = -\lambda^n e_0^{n-1}ge_1 = \lambda^n z_{n,g}.$$

Since the algebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is freely generated by $(z_{n,g})_{(n,g)\in\mathbb{N}_{>0}\times G}$, it follows that $\lambda \bullet_{\mathcal{V}} (\widehat{\mathcal{W}}_G^{\mathrm{DR}}) \subset \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. Similarly, $(\lambda \bullet_{\mathcal{V}} -)^{-1}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}) \subset \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. Hence, $\lambda \bullet_{\mathcal{V}} (\widehat{\mathcal{W}}_G^{\mathrm{DR}}) = \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. This implies that $\lambda \bullet_{\mathcal{V}} -$ restricts to a $\mathbf{k}$-algebra automorphism $\lambda \bullet_{\mathcal{W}} -$ of

$\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ and that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\lambda \bullet_{\mathcal{W}} -} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{\lambda \bullet_{\mathcal{V}} -} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{1.22}$$

commutes. Let us show that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\lambda \bullet_{\mathcal{W}} -} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}} \\ (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes} 2} & \xrightarrow{(\lambda \bullet_{\mathcal{W}} -)^{\otimes 2}} & (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes} 2} \end{array} \tag{1.23}$$

commutes. Indeed, for $(n, g) \in \mathbb{N}_{>0} \times G$ we have

$$\begin{aligned} \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ (\lambda \bullet_{\mathcal{W}} -)(z_{n,g}) &= \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(\lambda^n z_{n,g}) = \lambda^n \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,g}) \\ &= \lambda^n z_{n,g} \otimes 1 + 1 \otimes \lambda^n z_{n,g} + \lambda^n \sum_{\substack{k=1 \\ h \in G}}^{n-1} z_{k,\phi(h)} \otimes z_{n-k,gh^{-1})} \\ &= (\lambda \bullet_{\mathcal{W}} -)^{\otimes 2} \Big(z_{n,g} \otimes 1 + 1 \otimes z_{n,g} + \sum_{\substack{k=1 \\ h \in G}}^{n-1} z_{k,h} \otimes z_{n-k,gh^{-1}} \Big) \\ &= (\lambda \bullet_{\mathcal{W}} -)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,g}). \end{aligned}$$

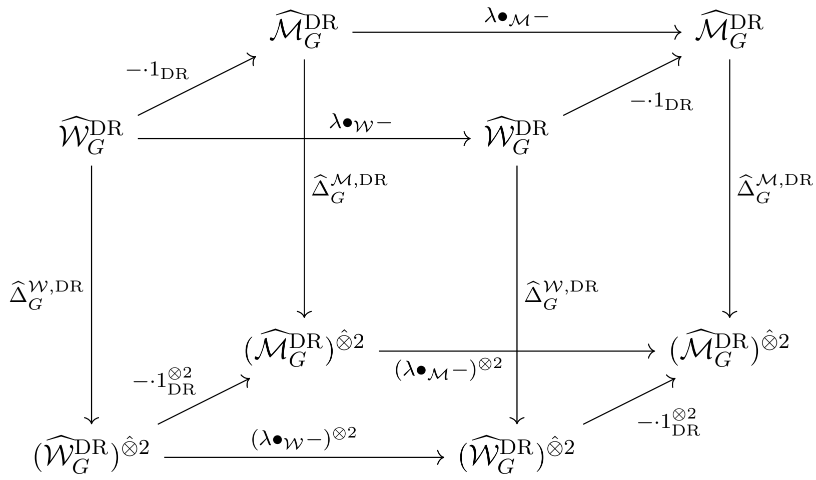
(ii) One checks that $\lambda \bullet_{\mathcal{V}} -$ preserves the submodule $\widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \sum_{g \in G} \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g-1)$. It follows that there is a unique $\mathbf{k}$-module automorphism $\lambda \bullet_{\mathcal{M}} -$ of $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ such that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{\lambda \bullet_{\mathcal{V}} -} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle - \cdot 1_{\mathrm{DR}}}\big\downarrow & & \big\downarrow{\scriptstyle - \cdot 1_{\mathrm{DR}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{\lambda \bullet_{\mathcal{M}} -} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{1.24}$$

commutes. Since $\lambda \bullet_{\mathcal{V}} -$ restricts to the automorphism $\lambda \bullet_{\mathcal{W}} -$ of $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$, we obtain the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\lambda \bullet_{\mathcal{W}} -} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle - \cdot 1_{\mathrm{DR}}}\big\downarrow & & \big\downarrow{\scriptstyle - \cdot 1_{\mathrm{DR}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{\lambda \bullet_{\mathcal{M}} -} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{1.25}$$

We then have the following cube



The left and the right faces are exactly the same square, which is commutative since it corresponds to Diagram 1.10. The upper side commutes thanks to Diagram (1.25) and the lower side is the tensor square of the upper side so is commutative. Finally, (i) gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of $- \cdot 1_{\mathrm{DR}}$ implies that the back side of the cube commutes, which proves that $\lambda \bullet_{\mathcal{M}} -$ is a coalgebra automorphism of $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$.

□

Proposition 1.3.6. *Let $\lambda \in \mathbf{k}^\times$.*

(i) The pair $(\lambda \bullet_{\mathcal{V}} -, \lambda \bullet_{\mathcal{M}} -)$ is an automorphism of $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ in the category $\mathbf{k}$-alg-mod$_{\mathrm{top}}$.

(ii) The pair $(\lambda \bullet_{\mathcal{W}} -, \lambda \bullet_{\mathcal{M}} -)$ is an automorphism of $((\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}), (\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}))$ in the category $\mathbf{k}$-HAMC$_{\mathrm{top}}$.

Proof.

(i) Let $(v, m) \in \widehat{\mathcal{V}}_G^{\mathrm{DR}} \times \widehat{\mathcal{M}}_G^{\mathrm{DR}}$. Since $- \cdot 1_{\mathrm{DR}} : \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is surjective, there exist $v' \in \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ such that $m = v' \cdot 1_{\mathrm{DR}}$. We have

$$\begin{aligned} \lambda \bullet_{\mathcal{M}} (vm) &= \lambda \bullet_{\mathcal{M}} (vv' \cdot 1_{\mathrm{DR}}) = (\lambda \bullet_{\mathcal{V}} vv') \cdot 1_{\mathrm{DR}} = (\lambda \bullet_{\mathcal{V}} v)\,(\lambda \bullet_{\mathcal{V}} v') \cdot 1_{\mathrm{DR}} \\ &= (\lambda \bullet_{\mathcal{V}} v)\,(\lambda \bullet_{\mathcal{M}} m), \end{aligned}$$

where the second and fourth equalities come from the commutativity of Diagram (1.24).

(ii) It follows from (i) and from Lemma 1.3.5.

□

1.3.3. *The torsor* $\mathsf{DMR}_\times^\iota(\mathbf{k})$.

Proposition 1.3.7. *For any $\lambda \in \mathbf{k}^\times$, the map $\lambda \bullet - : \mathbf{k}\langle\langle X \rangle\rangle \to \mathbf{k}\langle\langle X \rangle\rangle$ restricts to a group automorphism of $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$.*

Proof. It follows from Proposition 1.3.1 that $(\lambda \bullet -)_{|\,\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}$ is a group automorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$. It remains to prove that this permutation of $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ induces a permutation of its subset $\mathsf{DMR}_0^G(\mathbf{k})$. Let $\lambda \in \mathbf{k}^\times$ and $\Psi \in \mathsf{DMR}_0^G(\mathbf{k})$. Since $\lambda \bullet x_0 = \lambda x_0$ and $\lambda \bullet x_g = \lambda x_g$ for $g \in G$, Conditions (i), (iii), (iv) and (v) of Definition 1.2.1 are immediately satisfied by $\lambda \bullet \Psi$. In order to prove that Condition (ii) is satisfied, let us use Proposition 1.2.4. We have

$$\begin{aligned}(\lambda \bullet \Psi)^\star &= \left(\Gamma^{-1}_{\lambda\bullet\Psi}(-e_1)\beta(\lambda \bullet \Psi \otimes 1)\right) \cdot 1_{\mathrm{DR}} = \left((\lambda \bullet_{\mathcal{V}} \Gamma^{-1}_{\Psi}(-e_1))\,(\lambda \bullet_{\mathcal{V}} \beta(\Psi \otimes 1))\right) \cdot 1_{\mathrm{DR}} \\ &= \left(\lambda \bullet_{\mathcal{V}} (\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi \otimes 1))\right) \cdot 1_{\mathrm{DR}} = \lambda \bullet_{\mathcal{M}} \left(\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi \otimes 1) \cdot 1_{\mathrm{DR}}\right) \\ &= \lambda \bullet_{\mathcal{M}} \Psi^\star, \end{aligned} \tag{1.26}$$

where the second equality comes from the commutativity of Diagram (1.20) and Identity (1.21), the third one from the fact that $\lambda \bullet_{\mathcal{V}} -$ is an algebra morphism and the fourth one from Lemma 1.3.5. (ii). Therefore, we obtain that

$$\begin{aligned}\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}((\lambda \bullet \Psi)^\star) &= \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}(\lambda \bullet_{\mathcal{M}} \Psi^\star) = (\lambda \bullet_{\mathcal{M}} -)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}(\Psi^\star) \\ &= (\lambda \bullet_{\mathcal{M}} -)^{\otimes 2}((\Psi^\star)^{\otimes 2}) = (\lambda \bullet_{\mathcal{M}} \Psi^\star)^{\otimes 2} \\ &= (\lambda \bullet \Psi)^\star \otimes (\lambda \bullet \Psi)^\star,\end{aligned}$$

where the first and last equalities come from (1.26), the second one from Lemma 1.3.5. (ii) and the third one from Proposition 1.2.4 and the fact that $\Psi \in \mathsf{DMR}_0^G(\mathbf{k})$. This proves that $\lambda \bullet -$ restricts to a self-map of $\mathsf{DMR}_0^G(\mathbf{k})$. Following the same steps, one shows that $(\lambda \bullet -)^{-1} = \lambda^{-1} \bullet -$ restricts to a self-map of $\mathsf{DMR}_0^G(\mathbf{k})$ thus proving the statement. ☐

Proposition 1.3.7 enables us to state the following definition:

Definition 1.3.8. We denote $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ the semi-direct product of $\mathbf{k}^\times$ and $\mathsf{DMR}_0^G(\mathbf{k})$ with respect to the action given in Proposition 1.3.7. It is a subgroup of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Definition 1.3.9. Let $\iota : G \to \mathbb{C}^\times$ be a group embedding. We define

$$\mathsf{DMR}_\times^\iota(\mathbf{k}) := \{(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \,|\, \Psi \in \mathsf{DMR}_\lambda^\iota(\mathbf{k})\}. \tag{1.27}$$

Proposition 1.3.10. *Let $\iota : G \to \mathbb{C}^\times$ be a group embedding. The group $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ acts freely and transitively on $\mathsf{DMR}_\times^\iota(\mathbf{k})$ by left multiplication $\circledast$.*

In order to prove this, we will need the following Lemma:

Lemma 1.3.11. *Let $\iota : G \to \mathbb{C}^\times$ be a group embedding and $\lambda, \nu \in \mathbf{k}^\times$. If $\Phi \in \mathsf{DMR}_\nu^\iota(\mathbf{k})$ then $\lambda\nu^{-1} \bullet \Phi \in \mathsf{DMR}_\lambda^\iota(\mathbf{k})$.*

Proof. Let $\Phi \in \mathsf{DMR}_\nu^\iota(\mathbf{k})$. Since $\lambda\nu^{-1} \bullet x_0 = \lambda\nu^{-1} x_0$ and $\lambda\nu^{-1} \bullet x_g = \lambda\nu^{-1} x_g$ for $g \in G$, we have

- $(\lambda\nu^{-1} \bullet \Phi | x_0) = \lambda\nu^{-1}(\Phi|x_0) = 0$ and $(\lambda\nu^{-1} \bullet \Phi|x_1) = \lambda\nu^{-1}(\Phi|x_1) = 0$.
- $(\lambda\nu^{-1} \bullet \Phi|x_0x_1) = \lambda^2{\nu^{-1}}^2(\Phi|x_0x_1) = -\lambda^2{\nu^{-1}}^2\frac{\nu^2}{24} = -\frac{\lambda^2}{24}$.
- $(\lambda\nu^{-1} \bullet \Phi|x_{g_\iota} - x_{g_\iota^{-1}}) = \lambda\nu^{-1}(\Phi|x_{g_\iota} - x_{g_\iota^{-1}}) = \lambda\nu^{-1}\frac{|G|-2}{2}\nu = \frac{|G|-2}{2}\lambda$.

This proves respectively conditions (i), (iii) and (iv) of Definition 1.2.1. Condition (v) follows from condition (iv). Finally, one shows that Condition (ii) is satisfied by using the same arguments as in the proof of Proposition 1.3.7. □

Proof of Proposition 1.3.10. Since the action of the group $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on the set $\mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ by left multiplication $\circledast$ is free, so is its restriction to the action of $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ on $\mathsf{DMR}_\times^\iota(\mathbf{k})$. Let us show that this action is transitive. Let (λ,Ψ) and $(\nu,\Phi) \in \mathsf{DMR}_\times^\iota(\mathbf{k})$. Set $\mu = \lambda\nu^{-1}$. It follows from Lemma 1.3.11 that $\mu\bullet\Phi \in \mathsf{DMR}_\lambda^\iota(\mathbf{k})$. Thanks to Proposition 1.2.5.(ii), the action of the group $(\mathsf{DMR}_0^G(\mathbf{k}),\circledast)$ on $\mathsf{DMR}_\lambda^\iota(\mathbf{k})$ is transitive, therefore, there exists $\Lambda \in \mathsf{DMR}_0^G(\mathbf{k})$ such that $\Lambda \circledast (\mu \bullet \Phi) = \Psi$. Thus $(\mu,\Lambda) \in \mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ is such that

$$(\mu,\Lambda) \circledast (\nu,\Phi) = (\lambda,\Psi),$$

which proves the transitivity. □

1.4. **The torsor $\mathsf{DMR}_\times(\mathbf{k})$.**

1.4.1. *Action of the group* $\mathrm{Aut}(G)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle$. The group $\mathrm{Aut}(G)$ acts on $\mathbf{k}\langle\langle X\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms, the element $\phi \in \mathrm{Aut}(G)$ acting by the automorphism η_ϕ given by

$$x_0 \mapsto x_0, \quad x_g \mapsto x_{\phi(g)} \text{ for } g \in G. \tag{1.28}$$

One checks that, for any $\phi \in \mathrm{Aut}(G)$, the automorphism η_ϕ is a Hopf algebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$.

In addition, for any $\phi \in \mathrm{Aut}(G)$ and any $g \in G$, we have

$$\eta_\phi \circ t_g = t_g \circ \eta_\phi, \tag{1.29}$$

which can be verified by checking this identity on generators since both sides are given as a composition of $\mathbf{k}$-algebra morphisms. Let us show that

Proposition 1.4.1. *Let $\phi \in \mathrm{Aut}(G)$. The map η_ϕ restricts to a group automorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\circledast)$.*

In order to prove this, we will need the following Lemma:

Lemma 1.4.2. *Let $\phi \in \mathrm{Aut}(G)$, $g \in G$ and $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have*

$$\eta_\phi \circ \mathrm{aut}_\Psi = \mathrm{aut}_{\eta_\phi(\Psi)} \circ \eta_\phi. \tag{1.30}$$

Proof. Since all morphisms are algebra automorphisms of $\mathbf{k}\langle\langle X\rangle\rangle$, it is enough to check this identity on generators. We have

$$\eta_\phi \circ \mathrm{aut}_\Psi(x_0) = \eta_\phi(x_0) = x_0 = \mathrm{aut}_{\eta_\phi(\Psi)}(x_0) = \mathrm{aut}_{\eta_\phi(\Psi)} \circ \eta_\phi(x_0)$$

and for $g \in G$,

$$\begin{aligned}\eta_\phi \circ \mathrm{aut}_\Psi(x_g) &= \eta_\phi(\mathrm{Ad}_{t_g(\Psi^{-1})}(x_g)) = \mathrm{Ad}_{t_{\phi(g)}(\eta_\phi(\Psi)^{-1})}(\eta_\phi(x_g)) \\ &= \mathrm{Ad}_{t_{\phi(g)}(\eta_\phi(\Psi)^{-1})}(x_{\phi(g)}) = \mathrm{aut}_{\eta_\phi(\Psi)}(x_{\phi(g)}) = \mathrm{aut}_{\eta_\phi(\Psi)} \circ \eta_\phi(x_g),\end{aligned}$$

where the second equality comes from identity (1.29). □

Proof of Proposition 1.4.1. Let $\phi \in \mathrm{Aut}(G)$. Since η_ϕ is a Hopf algebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$, it restricts to a map $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Let $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have

$$\begin{aligned}\eta_\phi(\Psi \circledast \Phi) &= \eta_\phi(\Psi \mathrm{aut}_\Psi(\Phi)) = \eta_\phi(\Psi)\eta_\phi(\mathrm{aut}_\Psi(\Phi)) \\ &= \eta_\phi(\Psi)\mathrm{aut}_{\eta_\phi(\Psi)}(\eta_\phi(\Phi)) = \eta_\phi(\Psi) \circledast \eta_\phi(\Phi),\end{aligned}$$

where the third equality comes from Lemma 1.4.2. This proves that η_ϕ restricts to a group endomorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$. Finally, one has that $\eta_\phi^{-1} = \eta_{\phi^{-1}}$ and the above computations shows that η_ϕ^{-1} is an endomorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$, thus proving the statement. □

Lemma 1.4.3. *For* $(\phi, \lambda) \in \mathrm{Aut}(G) \times \mathbf{k}^\times$, *we have*

$$\eta_\phi \circ (\lambda \bullet -) = (\lambda \bullet -) \circ \eta_\phi.$$

Proof. Let $(\phi, \lambda) \in \mathrm{Aut}(G) \times \mathbf{k}^\times$. Since all the morphisms are algebra automorphisms of $\mathbf{k}\langle\langle X\rangle\rangle$, it is enough to check this identity on generators. We have

$$\eta_\phi \circ (\lambda \bullet x_0) = \eta_\phi(\lambda x_0) = \lambda x_0 = \lambda \bullet x_0 = \lambda \bullet \eta_\phi(x_0) = (\lambda \bullet -) \circ \eta_\phi(x_0)$$

and for $g \in G$,

$$\eta_\phi \circ (\lambda \bullet x_g) = \eta_\phi(\lambda x_g) = \lambda x_{\phi(g)} = \lambda \bullet x_{\phi(g)} = \lambda \bullet \eta_\phi(x_g) = (\lambda \bullet -) \circ \eta_\phi(x_g).$$

□

Propositions 1.3.1 and 1.4.1 and Lemma 1.4.3 enable us to define the following:

Definition 1.4.4. We denote $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ the semi-direct product of $\mathrm{Aut}(G) \times \mathbf{k}^\times$ and $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ with respect to the action given in Propositions 1.3.1 and 1.4.1. It consists of the set $\mathrm{Aut}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ endowed with a group law which will also be denoted $\circledast$ and we have for $(\phi, \lambda, \Psi), (\phi', \nu, \Phi) \in \mathrm{Aut}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$,

$$(\phi, \lambda, \Psi) \circledast (\phi', \nu, \Phi) := (\phi \circ \phi', \lambda\nu, \Psi \circledast \eta_\phi(\lambda \bullet \Phi)). \tag{1.31}$$

1.4.2. *Action of the group* $\mathrm{Aut}(G)$ *on* $\mathrm{Emb}(G)$. Let us denote

$$\mathrm{Emb}(G) := \{\iota : G \to \mathbb{C}^\times \mid \iota \text{ is a group embedding}\}. \tag{1.32}$$

Lemma 1.4.5. *The group* $\mathrm{Aut}(G)$ *acts freely and transitively on* $\mathrm{Emb}(G)$ *by*

$$(\phi, \iota) \longmapsto \iota \circ \phi^{-1}, \tag{1.33}$$

for $(\phi, \iota) \in \mathrm{Aut}(G) \times \mathrm{Emb}(G)$.

Proof. One knows that for any $\iota \in \mathrm{Emb}(G)$, $\iota(G) = \mu_N$. That gives rise to a group isomorphism $\widetilde{\iota} : G \to \mu_N(\mathbb{C})$. Therefore, for any $\iota, \iota' \in \mathrm{Emb}(G)$ there is a unique group automorphism $\phi = \widetilde{\iota'}^{-1} \circ \widetilde{\iota}$ of G such that $\iota \circ \phi^{-1} = \iota'$. □

Corollary 1.4.6. *The group* $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *acts freely and transitively on* $\mathrm{Emb}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *by*

$$(\phi, \lambda, \Psi) \cdot (\iota, \nu, \Phi) = (\iota \circ \phi^{-1}, \lambda\nu, \Psi \circledast \eta_\phi(\lambda \bullet \Phi)), \tag{1.34}$$

for $(\phi, \lambda, \Psi) \in (\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $(\iota, \nu, \Phi) \in \mathrm{Emb}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Proof. Let $(\iota,\nu,\Phi),(\iota',\nu',\Phi')\in \mathrm{Emb}(G)\times \mathbf{k}^\times\times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Thanks to Lemma 1.4.5, there is a unique $\phi\in\mathrm{Aut}(G)$ such that $\iota'=\iota\circ\phi^{-1}$. Set

$$\lambda=\nu^{-1}\nu' \text{ and } \Psi=\Phi'\circledast\eta_\phi(\lambda\bullet\Phi)^{\circledast(-1)}.$$

In conclusion, there is a unique $(\phi,\lambda,\Psi)\in(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ such that

$$(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)=(\iota',\nu',\Phi'),$$

which proves the statement. □

1.4.3. *Action of the group* $\mathrm{Aut}(G)$ *on crossed product algebras and module.* The group $\mathrm{Aut}(G)$ acts on $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by $\mathbf{k}$-algebra automorphisms the element $\phi\in\mathrm{Aut}(G)$ acting by the automorphism $\eta_\phi^{\mathcal{V}}$ given by

$$e_0\mapsto e_0,\quad e_1\mapsto e_1\ \text{ and } g\mapsto\phi(g)\ \text{ for } g\in G. \tag{1.35}$$

Lemma 1.4.7. *Let* $\phi\in\mathrm{Aut}(G)$. *The following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\ \eta_\phi\ } & \mathbf{k}\langle\langle X\rangle\rangle \\ {\scriptstyle \beta\circ(-\otimes 1)}\downarrow & & \downarrow{\scriptstyle \beta\circ(-\otimes 1)} \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{\ \eta_\phi^{\mathcal{V}}\ } & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{1.36}$$

commutes.

Proof. Since all arrows are $\mathbf{k}$-algebra morphisms, one easily checks the commutativity of generators. □

Lemma 1.4.8. *Let* $\phi\in\mathrm{Aut}(G)$.

(i) The $\mathbf{k}$*-algebra automorphism* $\eta_\phi^{\mathcal{V}}$ *of* $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ *restricts to a Hopf algebra automorphism* $\eta_\phi^{\mathcal{W}}$ *of* $(\widehat{\mathcal{W}}_G^{\mathrm{DR}},\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$.

(ii) The $\mathbf{k}$*-algebra automorphism* $\eta_\phi^{\mathcal{V}}$ *of* $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ *induces a coalgebra automorphism* $\eta_\phi^{\mathcal{M}}$ *of* $(\widehat{\mathcal{M}}_G^{\mathrm{DR}},\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$.

Proof.

(i) For $(n,g)\in\mathbb{N}_{>0}\times G$ we have

$$\eta_\phi^{\mathcal{V}}(z_{n,g})=\eta_\phi^{\mathcal{V}}(-e_0^{n-1}ge_1)=-e_0^{n-1}\phi(g)e_1=z_{n,\phi(g)}.$$

Since the algebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is freely generated by the family $(z_{n,g})_{(n,g)\in\mathbb{N}_{>0}\times G}$, it follows that $\eta_\phi^{\mathcal{V}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})\subset\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. Similarly, $(\eta_\phi^{\mathcal{V}})^{-1}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})\subset\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. Hence, $\eta_\phi^{\mathcal{V}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})=\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. This implies that $\eta_\phi^{\mathcal{V}}$ restricts to a $\mathbf{k}$-algebra automorphism of $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ which we denote $\eta_\phi^{\mathcal{W}}$ and that we have the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\ \eta_\phi^{\mathcal{W}}\ } & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{\ \eta_\phi^{\mathcal{V}}\ } & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{1.37}$$

Let us show that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\eta_\phi^{\mathcal{W}}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}}\downarrow & & \downarrow{\scriptstyle \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}} \\ (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\otimes 2} & \xrightarrow{(\eta_\phi^{\mathcal{W}})^{\otimes 2}} & (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\otimes 2} \end{array} \tag{1.38}$$

commutes. Indeed, for $(n, g) \in \mathbb{N}_{>0} \times G$ we have

$$\begin{aligned} \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ \eta_\phi^{\mathcal{W}}(z_{n,g}) &= \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,\phi(g)}) \\ &= z_{n,\phi(g)} \otimes 1 + 1 \otimes z_{n,\phi(g)} + \sum_{\substack{k=1 \\ h\in G}}^{n-1} z_{k,h} \otimes z_{n-k,\phi(g)h^{-1}} \\ &= z_{n,\phi(g)} \otimes 1 + 1 \otimes z_{n,\phi(g)} + \sum_{\substack{k=1 \\ h\in G}}^{n-1} z_{k,\phi(h)} \otimes z_{n-k,\phi(g)\phi(h^{-1})} \\ &= (\eta_\phi^{\mathcal{W}})^{\otimes 2}\Big(z_{n,g} \otimes 1 + 1 \otimes z_{n,g} + \sum_{\substack{k=1 \\ h\in G}}^{n-1} z_{k,h} \otimes z_{n-k,gh^{-1}}\Big) \\ &= (\eta_\phi^{\mathcal{W}})^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,g}). \end{aligned}$$

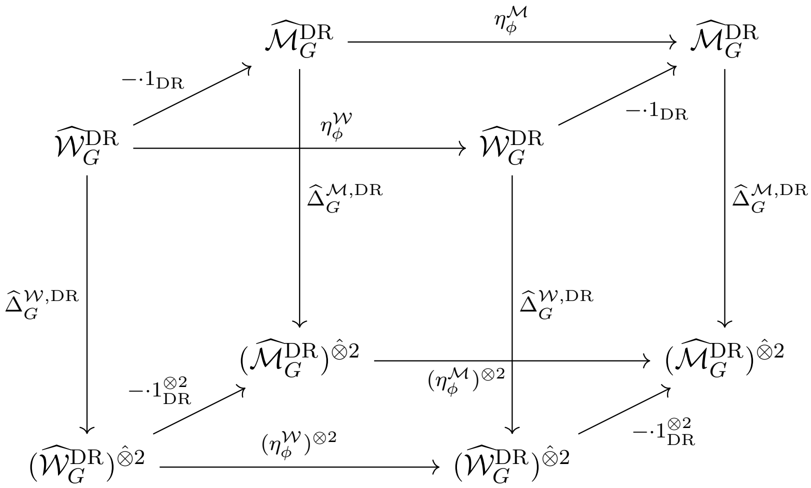
(ii) Let $\phi \in \mathrm{Aut}(G)$. One checks that $\eta_\phi^{\mathcal{V}}$ preserves the submodule $\hat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \sum_{g\in G} \hat{\mathcal{V}}_G^{\mathrm{DR}}(g-1)$. It follows that there is a unique $\mathbf{k}$-module automorphism $\eta_\phi^{\mathcal{M}}$ of $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ such that the following diagram

$$\begin{array}{ccc} \hat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{\eta_\phi^{\mathcal{V}}} & \hat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle -\cdot 1_{\mathrm{DR}}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{\eta_\phi^{\mathcal{M}}} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{1.39}$$

commutes. Combined with (i), it gives the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\eta_\phi^{\mathcal{W}}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle -\cdot 1_{\mathrm{DR}}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{\eta_\phi^{\mathcal{M}}} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{1.40}$$

We then have the following cube



The left and the right faces are exactly the same square, which is commutative since it corresponds to Diagram 1.10. The upper side commutes thanks to Diagram (1.40) and the lower side is the tensor square of the upper side so is commutative. Finally, (i) gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of $-\cdot 1_{\mathrm{DR}}$ implies that the back side of the cube commutes, which proves that $\eta_\phi^{\mathcal{M}}$ is a coalgebra automorphism of $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$. □

Proposition 1.4.9. *Let $\phi \in \mathrm{Aut}(G)$.*

(i) The pair $(\eta_\phi^{\mathcal{V}}, \eta_\phi^{\mathcal{M}})$ is an automorphism of $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ in the category $\mathbf{k}$-alg-mod$_{\mathrm{top}}$.

(ii) The pair $(\eta_\phi^{\mathcal{W}}, \eta_\phi^{\mathcal{M}})$ is an automorphism of $((\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}), (\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}))$ in the category $\mathbf{k}$-HAMC$_{\mathrm{top}}$.

Proof.

(i) Let $(v, m) \in \widehat{\mathcal{V}}_G^{\mathrm{DR}} \times \widehat{\mathcal{M}}_G^{\mathrm{DR}}$. Since $-\cdot 1_{\mathrm{DR}} : \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is surjective, there exist $v' \in \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ such that $m = v' \cdot 1_{\mathrm{DR}}$. We have

$$\begin{aligned}\eta_\phi^{\mathcal{M}}(vm) &= \eta_\phi^{\mathcal{M}}(vv' \cdot 1_{\mathrm{DR}}) = \eta_\phi^{\mathcal{V}}(vv') \cdot 1_{\mathrm{DR}} \\ &= \eta_\phi^{\mathcal{V}}(v)\, \eta_\phi^{\mathcal{V}}(v') \cdot 1_{\mathrm{DR}} = \eta_\phi^{\mathcal{V}}(v)\, \eta_\phi^{\mathcal{M}}(m),\end{aligned}$$

where the second and fourth equalities come from Lemma 1.4.8. (ii).

(ii) It follows from (i) and from Lemma 1.4.8.

□

1.4.4. *The torsor* $\mathsf{DMR}_\times(\mathbf{k})$. Lemma 1.4.5 sets up the following result:

Proposition 1.4.10. *Let $\lambda \in \mathbf{k}$. For $\iota, \iota' \in \mathrm{Emb}(G)$, the element $\phi \in \mathrm{Aut}(G)$ such that $\iota' = \iota \circ \phi$ is such that η_ϕ is a bijection between $\mathsf{DMR}_\lambda^\iota(\mathbf{k})$ and $\mathsf{DMR}_\lambda^{\iota'}(\mathbf{k})$.*

Proof. Since η_ϕ is a Hopf algebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$, it restricts to group automorphism of $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. It remains to show that $\eta_\phi : \mathsf{DMR}_\lambda^\iota(\mathbf{k}) \to \mathsf{DMR}_\lambda^{\iota\circ\phi^{-1}}(\mathbf{k})$

is a bijection. Let $\Psi \in \mathsf{DMR}^\iota_\lambda(\mathbf{k})$. Since $\phi(x_0) = x_0$ and $\phi(x_1) = x_1$, Conditions (i) and (iii) of Definition 1.2.1 are immediately satisfied by $\eta_\phi(\Psi)$. Moreover, since $g_{\iota\circ\phi^{-1}} = \phi(g_\iota)$, we have

$$\Big(\eta_\phi(\Psi)|x_{g_{\iota\circ\phi^{-1}}} - x_{g^{-1}_{\iota\circ\phi^{-1}}}\Big) = \Big(\eta_\phi(\Psi)|x_{\phi(g_\iota)} - x_{\phi(g_\iota^{-1})}\Big) = \Big(\eta_\phi(\Psi)|\eta_\phi(x_{g_\iota} - x_{g_\iota^{-1}})\Big)$$
$$= \Big(\Psi|x_{g_\iota} - x_{g_\iota^{-1}}\Big) = \frac{|G|-2}{2}\lambda.$$

Then Identity (iv) of Definition 1.2.1 follows. One checks Identity (v) in a similar way. Let us prove that Condition (ii) is satisfied by $\eta_\phi(\Psi)$. We have

$$\begin{aligned}(\eta_\phi(\Psi))^\star &= \Big(\Gamma^{-1}_{\eta_\phi(\Psi)}(-e_1)\beta(\eta_\phi(\Psi)\otimes 1)\Big)\cdot 1_{\mathrm{DR}} = \big(\Gamma^{-1}_{\Psi}(-e_1)\beta(\eta_\phi(\Psi)\otimes 1)\big)\cdot 1_{\mathrm{DR}}\\ &= \big(\Gamma^{-1}_{\Psi}(-e_1)\eta^{\mathcal{V}}_\phi(\beta(\Psi\otimes 1))\big)\cdot 1_{\mathrm{DR}} = \big(\eta^{\mathcal{V}}_\phi\left(\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi\otimes 1)\right)\big)\cdot 1_{\mathrm{DR}}\\ &= \eta^{\mathcal{M}}_\phi\left(\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi\otimes 1)\cdot 1_{\mathrm{DR}}\right) = \eta^{\mathcal{M}}_\phi(\Psi^\star),\end{aligned}$$

where the second equality comes from the fact that $\eta_\phi(x_1) = x_1$, the third one from the commutativity of Diagram (1.36), the fourth one from the fact that $\eta^{\mathcal{V}}_\phi(e_1) = e_1$ and the fifth one from the commutativity of Diagram (1.39). Therefore, thanks to Lemma 1.4.8.(ii) and the fact that $\Psi \in \mathsf{DMR}^\iota_\lambda(\mathbf{k})$, we obtain that

$$\begin{aligned}\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G((\eta_\phi(\Psi))^\star) &= \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\left(\eta^{\mathcal{M}}_\phi(\Psi^\star)\right) = (\eta^{\mathcal{M}}_\phi)^{\otimes 2}\left(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G(\Psi^\star)\right)\\ &= (\eta^{\mathcal{M}}_\phi)^{\otimes 2}(\Psi^\star\otimes\Psi^\star) = \eta^{\mathcal{M}}_\phi(\Psi^\star)^{\otimes 2} = (\eta_\phi(\Psi))^\star\otimes(\eta_\phi(\Psi))^\star,\end{aligned}$$

which implies, by Proposition 1.2.4, that condition (ii) of Definition 1.2.1 is verified by $(\eta_\phi(\Psi))^\star$. This proves that η_ϕ restricts to a map $\mathsf{DMR}^\iota_\lambda(\mathbf{k}) \to \mathsf{DMR}^{\iota\circ\phi^{-1}}_\lambda(\mathbf{k})$. Finally, following the same steps, one shows that $\eta_\phi^{-1} = \eta_{\phi^{-1}}$ restricts to a map $\mathsf{DMR}^{\iota\circ\phi^{-1}}_\lambda(\mathbf{k}) \to \mathsf{DMR}^\iota_\lambda(\mathbf{k})$ thus proving the statement. □

Corollary 1.4.11. *For any $\phi \in \mathrm{Aut}(G)$, the map η_ϕ restricts to a group automorphism of $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$.*

Proof. From Proposition 1.4.10 it follows that for $\lambda = 0$ and any $\phi \in \mathrm{Aut}(G)$ the map η_ϕ restricts to a bijection from $\mathsf{DMR}^G_0(\mathbf{k})$ to itself. In addition, since $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$ is a subgroup of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$, Proposition 1.4.1 states that this map is a group morphism. □

Proposition 1.3.7, Corollary 1.4.11 and Lemma 1.4.3 enable us to state the following definition:

Definition 1.4.12. We denote $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}^G_0(\mathbf{k})$ the semi-direct product of $\mathrm{Aut}(G)\times\mathbf{k}^\times$ and $\mathsf{DMR}^G_0(\mathbf{k})$ with respect to the group action of $\mathrm{Aut}(G)\times\mathbf{k}^\times$ induced by Corollary 1.4.11 and Proposition 1.3.7. It is a subgroup of $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Definition 1.4.13. We define

$$\mathsf{DMR}_\times(\mathbf{k}) := \{(\iota,\lambda,\Psi)\in\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)\,|\,(\lambda,\Psi)\in\mathsf{DMR}^\iota_\times(\mathbf{k})\}. \tag{1.41}$$

Proposition 1.4.14. *The group* $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}_0^G(\mathbf{k})$ *acts freely and transitively on* $\mathsf{DMR}_\times(\mathbf{k})$ *by*

$$(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)=(\iota\circ\phi^{-1},\lambda\nu,\Psi\circledast\eta_\phi(\lambda\bullet\Phi)), \tag{1.42}$$

for $(\phi,\lambda,\Psi)\in\mathrm{Aut}(G)\times\mathbf{k}^\times\times\mathsf{DMR}_0^G(\mathbf{k})$ *and* $(\iota,\nu,\Phi)\in\mathsf{DMR}_\times(\mathbf{k})$.

Proof. Let $(\iota,\nu,\Phi),(\iota',\nu',\Phi')\in\mathsf{DMR}_\times(\mathbf{k})$. Thanks to Lemma 1.4.5, there is a unique $\phi\in\mathrm{Aut}(G)$ such that $\iota'=\iota\circ\phi^{-1}$. Set $\lambda=\nu^{-1}\nu'$. Since $\Phi\in\mathsf{DMR}_\nu^\iota(\mathbf{k})$, thanks to Lemma 1.3.11 and Proposition 1.4.10, it follows that $\eta_\phi(\lambda\bullet\Phi)\in\mathsf{DMR}_{\nu'}^{\iota'}(\mathbf{k})$. Thanks to Proposition 1.2.5.(ii), the set $\mathsf{DMR}_{\nu'}^{\iota'}(\mathbf{k})$ is a torsor for the action of the group $(\mathsf{DMR}_0^G(\mathbf{k}),\circledast)$. Therefore, there is a unique $\Psi\in\mathsf{DMR}_0^G(\mathbf{k})$ such that $\Psi\circledast\eta_\phi(\lambda\bullet\Phi)=\Phi'$. In conclusion, there is a unique $(\phi,\lambda,\Psi)\in(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}_0^G(\mathbf{k})$ such that

$$(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)=(\iota',\nu',\Phi'),$$

which proves the statement. □

Corollary 1.4.15. *The pair* $\Big((\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}_0^G(\mathbf{k}),\mathsf{DMR}_\times(\mathbf{k})\Big)$ *is a subtorsor of* $\Big((\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)\Big)$.

Proof. It follows from Propositions 1.4.6 and 1.4.14. □

2. The double shuffle group as a stabilizer of a "de Rham" coproduct

In this section, we recall the action of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\circledast)$ on the algebra-module $\big(\widehat{\mathcal{W}}_G^{\mathrm{DR}},\widehat{\mathcal{M}}_G^{\mathrm{DR}}\big)$ given in [Yad]. This action enables us in §2.1, to construct an action of the group $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on the algebra-module $\big(\widehat{\mathcal{W}}_G^{\mathrm{DR}},\widehat{\mathcal{M}}_G^{\mathrm{DR}}\big)$. This leads us in §2.2 to define the stabilizer groups of the coproducts $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ and $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$. These stabilizers are related to stabilizers arising from the action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\circledast)$, which contain $\mathsf{DMR}_0^G(\mathbf{k})$ thanks to [EF0]. Thanks to the main result of [Yad], we conclude in Corollary 2.2.5 a chain of inclusions involving the former stabilizers and $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}_0^G(\mathbf{k})$.

2.1. Group actions on the algebra-module $\big(\widehat{\mathcal{W}}_G^{\mathrm{DR}},\widehat{\mathcal{M}}_G^{\mathrm{DR}}\big)$.

2.1.1. *Actions of the group* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\circledast)$. For $\Psi\in\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, there is a unique topological $\mathbf{k}$-algebra automorphism ${}^\Gamma\mathrm{aut}_\Psi^{\mathcal{V},(1)}$ of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ such that ([Yad, Definition 2.3.1]

$$\begin{aligned} e_0&\mapsto\Gamma_\Psi^{-1}(-e_1)\beta(\Psi\otimes1)\,e_0\,\beta(\Psi^{-1}\otimes1)\Gamma_\Psi(-e_1)\\ e_1&\mapsto\Gamma_\Psi^{-1}(-e_1)\,e_1\,\Gamma_\Psi(-e_1)\\ g&\mapsto\Gamma_\Psi^{-1}(-e_1)\beta(\Psi\otimes1)\,g\,\beta(\Psi^{-1}\otimes1)\Gamma_\Psi(-e_1).\end{aligned} \tag{2.1}$$

This automorphism restricts to a topological $\mathbf{k}$-algebra automorphism ${}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}$ of $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ ([Yad, Proposition-Definition 2.3.2]). It is such that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{2.2}$$

commutes. Next, one defines the topological $\mathbf{k}$-module automorphism ${}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}$ of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by

$$ {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(10)} := {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)} \circ r_{\Gamma_{\Psi}^{-1}(-e_1)\beta(\Psi\otimes 1)}. \tag{2.3}$$

This automorphism induces a topological $\mathbf{k}$-module automorphism ${}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}$ of $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ such that the following diagram ([Yad, Definition 2.3.4])

$$\begin{array}{ccc} \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle -\cdot 1_{\mathrm{DR}}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{2.4}$$

commutes.

Lemma 2.1.1 ([Yad, Lemma 2.3.5]). *For any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the following pairs are automorphisms in the category* $\mathbf{k}$-alg-mod$_{\mathrm{top}}$*:*

(i) $\left({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}\right)$ *is an automorphism of* $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}})$.

(ii) $\left({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}\right)$ *is an automorphism of* $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$.

(iii) $\left({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}\right)$ *is an automorphism of* $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$.

The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ acts on $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by ([Yad, Proposition 2.3.3])

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-alg}}^{\mathrm{top}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}}); \quad \Psi \longmapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}. \tag{2.5}$$

Thanks to this and the commutativity of Diagram (2.2), the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ acts on $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ by ([Yad, Proposition 2.3.3])

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-alg}}^{\mathrm{top}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}); \quad \Psi \longmapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}. \tag{2.6}$$

On the other hand, the action (2.5) induces an action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-mod}}^{\mathrm{top}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}}); \quad \Psi \longmapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}. \tag{2.7}$$

Thanks to the commutativity of Diagram (2.4), the action action (2.7) induces an action of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ by ([Yad, Proposition 2.3.6])

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-mod}}^{\mathrm{top}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}}); \quad \Psi \longmapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}. \tag{2.8}$$

Proposition 2.1.2.

(i) The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *acts on* $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}})$ *by*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}^{\mathrm{top}}_{\mathbf{k}\text{-alg-mod}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}}); \quad \Psi \longmapsto ({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}).$$

(ii) The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *acts on* $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ *by*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}^{\mathrm{top}}_{\mathbf{k}\text{-alg-mod}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}); \quad \Psi \longmapsto ({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}).$$

(iii) The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *acts on* $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ *by*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}^{\mathrm{top}}_{\mathbf{k}\text{-alg-mod}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}); \quad \Psi \longmapsto ({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}).$$

Proof. This follows from Lemma 2.1.1 and the fact that (2.5) – (2.8) define actions. □

2.1.2. *Actions of the group* $\mathbf{k}^{\times} \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Definition 2.1.3. For $(\lambda, \Psi) \in \mathbf{k}^{\times} \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define the topological $\mathbf{k}$-algebra-module automorphism $\left({}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{V},(1)}, {}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{V},(10)}\right)$ of $\left(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}}\right)$ given by

$$\left({}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{V},(1)}, {}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{V},(10)}\right) := \left({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}\right) \circ \big((\lambda \bullet_{\mathcal{V}} -), (\lambda \bullet_{\mathcal{V}} -)\big),$$

with $(\lambda \bullet_{\mathcal{V}} -) \in \mathrm{Aut}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}})$ given in (1.19).

Proposition-Definition 2.1.4. *For* $(\lambda, \Psi) \in \mathbf{k}^{\times} \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, we define the topological* $\mathbf{k}$*-algebra-module automorphism* $\left({}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{W},(1)}, {}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)}\right)$ *of* $\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}\right)$ *given by*

$$\left({}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{W},(1)}, {}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)}\right) := \left({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}\right) \circ \big((\lambda \bullet_{\mathcal{W}} -), (\lambda \bullet_{\mathcal{M}} -)\big).$$

It is such that the following diagrams

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{W},(1)}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{V},(1)}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{2.9}$$

and

$$\begin{array}{ccc} \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{V},(10)}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle -\cdot 1_{\mathrm{DR}}}\big\downarrow & & \big\downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)}} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{2.10}$$

commute.

Proof. From Propositions 1.3.6.(ii) and 2.1.2.(iii) we have that the pairs $(\lambda \bullet_{\mathcal{W}} -, \lambda \bullet_{\mathcal{W}} -)$ and $\left({}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}, {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{M},(10)}\right)$ are morphisms in $\mathbf{k}$-alg-mod$_{\mathrm{top}}$; the composition is then a morphism in $\mathbf{k}$-alg-mod$_{\mathrm{top}}$. Next, the commutativity of the diagrams follows from the commutativity of Diagrams (1.22) and (2.2) and Diagrams (1.24) and (2.4). □

Lemma 2.1.5. *For* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, we have*

$$\,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi} \circ (\lambda \bullet_{\mathcal{V}} -) = (\lambda \bullet_{\mathcal{V}} -) \circ \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}. \tag{2.11}$$

Proof. Since both sides are given as a composition of $\mathbf{k}$-algebra morphisms of $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$, it is enough to verify this identity on generators. We have

$$\begin{aligned}
\,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(\lambda \bullet_{\mathcal{V}} e_0) &= \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(\lambda e_0) = \lambda \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(e_0) \\
&= \lambda\, \Gamma^{-1}_{\lambda\bullet\Psi}(-e_1)\beta(\lambda \bullet \Psi \otimes 1)\, e_0\, \beta(\lambda \bullet \Psi^{-1} \otimes 1)\Gamma_{\lambda\bullet\Psi}(-e_1) \\
&= \big(\lambda \bullet_{\mathcal{V}} \Gamma^{-1}_{\Psi}(-e_1)\big)\, (\lambda \bullet_{\mathcal{V}} \beta(\Psi \otimes 1))\, \lambda\, e_0\, \big(\lambda \bullet_{\mathcal{V}} \beta(\Psi^{-1} \otimes 1)\big)\, (\lambda \bullet \Gamma_\Psi(-e_1)) \\
&= \lambda \bullet_{\mathcal{V}} \big(\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi \otimes 1)\, e_0\, \beta(\Psi^{-1} \otimes 1)\Gamma_\Psi(-e_1)\big) \\
&= \lambda \bullet_{\mathcal{V}} \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}(e_0),
\end{aligned}$$

where the fourth equality comes from the commutativity of Diagram (1.20) and Identity (1.21) and the fifth one from the fact that $\lambda \bullet_{\mathcal{V}} -$ is an algebra morphism. Next,

$$\begin{aligned}
\,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(\lambda \bullet_{\mathcal{V}} e_1) &= \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(\lambda e_1) = \lambda \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(e_1) = \lambda\, \Gamma^{-1}_{\lambda\bullet\Psi}(-e_1)\, e_1\, \Gamma_{\lambda\bullet\Psi}(-e_1) \\
&= \big(\lambda \bullet_{\mathcal{V}} \Gamma^{-1}_{\Psi}(-e_1)\big)\ \lambda\, e_1\ (\lambda \bullet \Gamma_\Psi(-e_1)) \\
&= \lambda \bullet_{\mathcal{V}} \big(\Gamma^{-1}_{\Psi}(-e_1)\, e_1\, \Gamma_\Psi(-e_1)\big) \\
&= \lambda \bullet_{\mathcal{V}} \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}(e_1),
\end{aligned}$$

where the fourth equality comes from Identity (1.21) and the fifth one from the fact that $\lambda \bullet_{\mathcal{V}} -$ is an algebra morphism. Finally, for $g \in G$,

$$\begin{aligned}
\,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(\lambda \bullet_{\mathcal{V}} g) &= \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(g) = \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\lambda\bullet\Psi}(g) \\
&= \Gamma^{-1}_{\lambda\bullet\Psi}(-e_1)\beta(\lambda \bullet \Psi \otimes 1)\, g\, \beta(\lambda \bullet \Psi^{-1} \otimes 1)\Gamma_{\lambda\bullet\Psi}(-e_1) \\
&= \big(\lambda \bullet_{\mathcal{V}} \Gamma^{-1}_{\Psi}(-e_1)\big)\, (\lambda \bullet_{\mathcal{V}} \beta(\Psi \otimes 1))\ g\ \big(\lambda \bullet_{\mathcal{V}} \beta(\Psi^{-1} \otimes 1)\big)\, (\lambda \bullet \Gamma_\Psi(-e_1)) \\
&= \lambda \bullet_{\mathcal{V}} \big(\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi \otimes 1)\, g\, \beta(\Psi^{-1} \otimes 1)\Gamma_\Psi(-e_1)\big) \\
&= \lambda \bullet_{\mathcal{V}} \,^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}(g),
\end{aligned}$$

where the fourth equality comes from the commutativity of Diagram (1.20) and Identity (1.21) and the fifth one from the fact that $\lambda \bullet_{\mathcal{V}} -$ is an algebra morphism. □

Corollary 2.1.6. *For* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, we have*

(i) $\,^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{\lambda\bullet\Psi} \circ (\lambda \bullet_{\mathcal{W}} -) = (\lambda \bullet_{\mathcal{W}} -) \circ \,^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{\Psi}$.
(ii) $\,^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{\lambda\bullet\Psi} \circ (\lambda \bullet_{\mathcal{M}} -) = (\lambda \bullet_{\mathcal{M}} -) \circ \,^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}$.

Proof.

(i) This follows from Lemma 2.1.5 thanks to Lemma 1.3.5.(i) and to the commutativity of Diagram (2.2).
(ii) This follows from Lemma 2.1.5 thanks to Lemma 1.3.5.(ii) and to the commutativity of Diagram (2.4).

□

Corollary 2.1.7. *The group* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *acts on* $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ *by*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{top}}_{\mathbf{k}\text{-alg-mod}}\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}\right); (\lambda, \Psi) \longmapsto \left({}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right).$$

Proof. Let $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have

$$\begin{aligned}
{}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} &= {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda\nu,\Psi\circledast\lambda\bullet\Phi)} = {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi\circledast\lambda\bullet\Phi} \circ (\lambda\nu \bullet_{\mathcal{V}} -) \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\lambda\bullet\Phi} \circ (\lambda \bullet_{\mathcal{W}} -) \circ (\nu \bullet_{\mathcal{W}} -) \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi} \circ (\lambda \bullet_{\mathcal{W}} -) \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Phi} \circ (\nu \bullet_{\mathcal{W}} -) \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\Phi)},
\end{aligned}$$

where the third equality comes from the fact that $\Psi \mapsto {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}$ and $\lambda \mapsto (\lambda \bullet_{\mathcal{V}} -)$ are group actions and the fourth one from Corollary 2.1.6.(i). Next, we have

$$\begin{aligned}
{}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} &= {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda\nu,\Psi\circledast\lambda\bullet\Phi)} = {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi\circledast\lambda\bullet\Phi} \circ (\lambda\nu \bullet_{\mathcal{M}} -) \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\lambda\bullet\Phi} \circ (\lambda \bullet_{\mathcal{M}} -) \circ (\nu \bullet_{\mathcal{M}} -) \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(1)}_{\Psi} \circ (\lambda \bullet_{\mathcal{M}} -) \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Phi} \circ (\nu \bullet_{\mathcal{M}} -) \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\nu,\Phi)},
\end{aligned}$$

where the third equality comes from the fact that $\Psi \mapsto {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}$ and $\lambda \mapsto (\lambda \bullet_{\mathcal{V}} -)$ are group actions and the fourth one from Corollary 2.1.6.(ii). □

2.1.3. *Actions of the group* $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Definition 2.1.8. For $(\phi, \lambda, \Psi) \in \mathrm{Aut}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define the topological $\mathbf{k}$-algebra-module automorphism $\left({}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\phi,\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\phi,\lambda,\Psi)}\right)$ of $\left(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}}\right)$ given by

$$\left({}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\phi,\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\phi,\lambda,\Psi)}\right) := \left({}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}\right) \circ \left(\eta^{\mathcal{V}}_{\phi}, \eta^{\mathcal{V}}_{\phi}\right),$$

with $\eta^{\mathcal{V}}_{\phi} \in \mathrm{Aut}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}})$ given in (1.35).

Proposition-Definition 2.1.9. *For* $(\phi, \lambda, \Psi) \in \mathrm{Aut}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, we define the topological* $\mathbf{k}$*-algebra-module automorphism* $\left({}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)$ *of* $\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}\right)$ *given by*

$$\left({}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right) := \left({}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right) \circ \left(\eta^{\mathcal{W}}_{\phi}, \eta^{\mathcal{M}}_{\phi}\right).$$

It is such that the following diagrams

$$\begin{array}{ccc}
\widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\
\big\downarrow & & \big\downarrow \\
\widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{{}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\phi,\lambda,\Psi)}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}}
\end{array} \tag{2.12}$$

and

$$
(2.13)\qquad
\begin{array}{ccc}
\widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{\ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\phi,\lambda,\Psi)}\ } & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\
{\scriptstyle -\cdot 1_{\mathrm{DR}}}\big\downarrow & & \big\downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\
\widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow{\ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\ } & \widehat{\mathcal{M}}_G^{\mathrm{DR}}
\end{array}
$$

commute.

Proof. From Proposition 1.4.9.(ii) and Proposition-Definition 2.1.4, we have that the pairs $(\eta_\phi^{\mathcal{W}}, \eta_\phi^{\mathcal{M}})$ and $\left({}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}, {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)$ are morphisms in $\mathbf{k}$-alg-mod$_{\mathrm{top}}$; the composition is then a morphism in $\mathbf{k}$-alg-mod$_{\mathrm{top}}$. Next, the commutativity of the diagrams follows from the commutativity of Diagrams (1.37) and (2.9) and Diagrams (1.39) and (2.10). □

Lemma 2.1.10. *For $(\phi, \lambda, \Psi) \in \mathrm{Aut}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have*

$$
{}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\eta_\phi(\Psi))} = \eta_\phi^{\mathcal{V}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (\eta_\phi^{\mathcal{V}})^{-1}.
$$

Proof. Since both sides are given as composition of $\mathbf{k}$-algebra morphisms of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$, it is enough to verify this identity on generators. We have

$$
\begin{aligned}
{}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\eta_\phi(\Psi))}(e_0) &= \mathrm{Ad}_{\Gamma^{-1}_{\eta_\phi(\Psi)}(-e_1)\beta(\eta_\phi(\Psi)\otimes 1)}(\lambda e_0) = \mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)\eta_\phi^{\mathcal{V}}(\beta(\Psi\otimes 1))}(\lambda e_0) \\
&= \eta_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi\otimes 1)}\big(\lambda(\eta_\phi^{\mathcal{V}})^{-1}(e_0)\big)\right) = \eta_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi\otimes 1)}(\lambda e_0)\right) \\
&= \eta_\phi^{\mathcal{V}}\left({}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(e_0)\right) = \eta_\phi^{\mathcal{V}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (\eta_\phi^{\mathcal{V}})^{-1}(e_0),
\end{aligned}
$$

where the second equality comes from the identity $\Gamma_{\eta_\phi(\Psi)}(-e_1) = \Gamma_\Psi(-e_1)$ and from the commutativity of Diagram (1.36) and the third one from the fact that $\eta_\phi^{\mathcal{V}}$ is an algebra morphism and from the equality $\eta_\phi^{\mathcal{V}}(\Gamma_\Psi(-e_1)) = \Gamma_\Psi(-e_1)$. Next,

$$
\begin{aligned}
{}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\eta_\phi(\Psi))}(e_1) &= \mathrm{Ad}_{\Gamma^{-1}_{\eta_\phi(\Psi)}(-e_1)}(\lambda e_1) = \mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)}(\lambda e_1) \\
&= \eta_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)}\big(\lambda(\eta_\phi^{\mathcal{V}})^{-1}(e_1)\big)\right) = \eta_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)}(\lambda e_1)\right) \\
&= \eta_\phi^{\mathcal{V}}\left({}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(e_1)\right) = \eta_\phi^{\mathcal{V}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (\eta_\phi^{\mathcal{V}})^{-1}(e_1),
\end{aligned}
$$

where the second equality comes from the identity $\Gamma_{\eta_\phi(\Psi)}(-e_1) = \Gamma_\Psi(-e_1)$ and the third one from the fact that $\eta_\phi^{\mathcal{V}}$ is an algebra morphism and from the equality $\eta_\phi^{\mathcal{V}}(\Gamma_\Psi(-e_1)) = \Gamma_\Psi(-e_1)$. Finally, for $g \in G$,

$$
\begin{aligned}
{}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\eta_\phi(\Psi))}(g) &= \mathrm{Ad}_{\Gamma^{-1}_{\eta_\phi(\Psi)}(-e_1)\beta(\eta_\phi(\Psi)\otimes 1)}(g) = \mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)\eta_\phi^{\mathcal{V}}(\beta(\Psi\otimes 1))}(g) \\
&= \eta_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi\otimes 1)}\big((\eta_\phi^{\mathcal{V}})^{-1}(g)\big)\right) = \eta_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma^{-1}_{\Psi}(-e_1)\beta(\Psi\otimes 1)}(\phi^{-1}(g))\right) \\
&= \eta_\phi^{\mathcal{V}}\left({}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(\phi^{-1}(g))\right) = \eta_\phi^{\mathcal{V}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (\eta_\phi^{\mathcal{V}})^{-1}(g),
\end{aligned}
$$

where the second equality comes from the identity $\Gamma_{\eta_\phi(\Psi)}(-e_1) = \Gamma_\Psi(-e_1)$ and from the commutativity of Diagram (1.36), the third one from the fact that $\eta_\phi^{\mathcal{V}}$ is an algebra morphism and from the equality $\eta_\phi^{\mathcal{V}}(\Gamma_\Psi(-e_1)) = \Gamma_\Psi(-e_1)$ and the fourth and sixth ones from the fact that $(\eta_\phi^{\mathcal{V}})^{-1}(g) = \phi^{-1}(g)$. □

Corollary 2.1.11. *For* $(\phi, \lambda, \Psi) \in \mathrm{Aut}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *we have*

(i) ${}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\eta_\phi(\Psi))} = \eta_\phi^{\mathcal{W}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ (\eta_\phi^{\mathcal{W}})^{-1}$.

(ii) ${}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\eta_\phi(\Psi))} = \eta_\phi^{\mathcal{M}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ (\eta_\phi^{\mathcal{M}})^{-1}$.

Proof. This follows from Lemma 2.1.10 thanks to Proposition-Definition 2.1.4 and Lemma 1.4.8. □

Corollary 2.1.12. *The group* $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *acts on* $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ *by*

$$(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}); (\phi,\lambda,\Psi) \mapsto \left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}, {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)$$

Proof. Let $(\phi, \lambda, \Psi), (\phi', \nu, \Phi) \in (\mathrm{Aut}(G) \times \mathbf{k}^\times) \rtimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have

$$\begin{aligned}
{}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)\circledast(\phi',\nu,\Phi)} &= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi\circ\phi',\lambda\nu,\Psi\circledast(\eta_\phi(\lambda\bullet\Phi)))} = {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda\nu,\Psi\circledast(\lambda\bullet\eta_\phi(\Phi)))} \circ \eta^{\mathcal{W}}_{\phi\circ\phi'} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)\circledast(\nu,\eta_\phi(\Phi))} \circ \eta_\phi^{\mathcal{W}} \circ \eta_{\phi'}^{\mathcal{W}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,(\eta_\phi(\Phi)))} \circ \eta_\phi^{\mathcal{W}} \circ \eta_{\phi'}^{\mathcal{W}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ \eta_\phi^{\mathcal{W}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\Phi)} \circ (\eta_\phi^{\mathcal{W}})^{-1} \circ \eta_\phi^{\mathcal{W}} \circ \eta_{\phi'}^{\mathcal{W}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ \eta_\phi^{\mathcal{W}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\Phi)} \circ \eta_{\phi'}^{\mathcal{W}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi',\nu,\Phi)},
\end{aligned}$$

where the second equality comes from Lemma 1.4.3, the third one from the fact that $\eta^{\mathcal{W}} : \mathrm{Aut}(G) \to \mathrm{Aut}^{\mathrm{top}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})$ is a group morphism, the fourth one from Corollary 2.1.7 and the fifth one from Corollary 2.1.11.(i). Next, we have

$$\begin{aligned}
{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)\circledast(\phi',\nu,\Phi)} &= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi\circ\phi',\lambda\nu,\Psi\circledast(\eta_\phi(\lambda\bullet\Phi)))} = {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda\nu,\Psi\circledast(\lambda\bullet\eta_\phi(\Phi)))} \circ \eta^{\mathcal{M}}_{\phi\circ\phi'} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)\circledast(\nu,\eta_\phi(\Phi))} \circ \eta_\phi^{\mathcal{M}} \circ \eta_{\phi'}^{\mathcal{M}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\nu,(\eta_\phi(\Phi)))} \circ \eta_\phi^{\mathcal{M}} \circ \eta_{\phi'}^{\mathcal{M}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \eta_\phi^{\mathcal{M}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\nu,\Phi)} \circ (\eta_\phi^{\mathcal{M}})^{-1} \circ \eta_\phi^{\mathcal{M}} \circ \eta_{\phi'}^{\mathcal{M}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \eta_\phi^{\mathcal{M}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\nu,\Phi)} \circ \eta_{\phi'}^{\mathcal{M}} \\
&= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi',\nu,\Phi)},
\end{aligned}$$

where the second equality comes from Lemma 1.4.3, the third one from the fact that $\eta^{\mathcal{M}} : \mathrm{Aut}(G) \to \mathrm{Aut}^{\mathrm{top}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}})$ is a group morphism, the fourth one from Corollary 2.1.7 and the fifth one from Corollary 2.1.11.(ii). □

2.2. **The double shuffle group as a stabilizer of a "de Rham" coproduct.**

Proposition 2.2.1.

(i) The group $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *acts on* $\mathrm{Cop}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})$ *by*

$$(\phi, \lambda, \Psi) \cdot D^{\mathcal{W}} := \left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\right)^{\otimes 2} \circ D^{\mathcal{W}} \circ \left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\right)^{-1}.$$

(ii) The group $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *acts on* $\mathrm{Cop}_{\mathbf{k}\text{-mod}_{\mathrm{top}}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}})$ *by*

$$(\phi, \lambda, \Psi) \cdot D^{\mathcal{M}} := \left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)^{\otimes 2} \circ D^{\mathcal{M}} \circ \left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)^{-1}.$$

Proof.

(i) This is the formula for the pull-back of the action (0.1) with $\mathcal{C} = \mathbf{k}\text{-alg}_{\mathrm{top}}$ and $O = \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ by the group morphism $(\phi, \lambda, \Psi) \mapsto {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}$ of Corollary 2.1.12.

(ii) This is the formula for the pull-back of the action (0.1) with $\mathcal{C} = \mathbf{k}\text{-mod}_{\mathrm{top}}$ and $O = \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ by the group morphism $(\phi, \lambda, \Psi) \mapsto {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}$ of Corollary 2.1.12.

□

Definition 2.2.2.

(i) We denote $\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \in \mathrm{Cop}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})$ for the action of Proposition 2.2.1.(i). Namely,

$$\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k}) :=$$
$$\left\{(\phi, \lambda, \Psi) \in (\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \,|\, \left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\right)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} = \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\right\}.$$

(ii) We denote $\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \in \mathrm{Cop}_{\mathbf{k}\text{-mod}_{\mathrm{top}}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}})$ for the action of Proposition 2.2.1.(ii). Namely,

$$\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k}) :=$$
$$\left\{(\phi, \lambda, \Psi) \in (\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \,|\, \left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} = \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right\}.$$

Since $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ is a subgroup of $(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the actions of Proposition 2.2.1 induce actions of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on the spaces $\mathrm{Cop}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})$ and $\mathrm{Cop}_{\mathbf{k}\text{-mod}_{\mathrm{top}}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}})$. This enables us to define the stabilizer subgroups (see [Yad, (2.29) and (2.31)])

$$\mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k}) := \left\{\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \,|\, \left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{\Psi}\right)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} = \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{\Psi}\right\}.$$

and

$$\mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k}) := \left\{\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \,|\, \left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}\right)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} = \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}\right\}.$$

Moreover, we have ([Yad, Theorem 2.4.1])

$$\mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \subset \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k}). \tag{2.14}$$

Proposition 2.2.3. *We have*

(i) $\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k}) = (\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k})$.

(ii) $\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) = (\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k})$.

It is a consequence of the following general lemma

Lemma 2.2.4. *Let us consider the semidirect product group $H \ltimes R$. If K is a subgroup of $H \ltimes R$ containing H, then*

$$K = H \ltimes (K \cap R).$$

Proof of Proposition 2.2.3. Set $\mathcal{X} = \mathcal{W}$ or $\mathcal{M}$. We use Lemma 2.2.4 where $H = \mathrm{Aut}(G) \times \mathbf{k}^\times$, $R = \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $K = \mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{X},\mathrm{DR}}_G)(\mathbf{k})$. We have that

$$K \cap R = \mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{X},\mathrm{DR}}_G)(\mathbf{k}) \cap \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) = \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{X},\mathrm{DR}}_G)(\mathbf{k}).$$

Additionally, $\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{X},\mathrm{DR}}_G)(\mathbf{k})$ contains $\mathrm{Aut}(G) \times \mathbf{k}^\times$. Therefore, the condition of Lemma 2.2.4 is met and the result then follows. □

Finally, one has from [EF0, Theorem 1.2] that

$$\mathsf{DMR}^G_0(\mathbf{k}) = \{\Psi \in \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \,|\, (\Psi|x_0) = (\Psi|x_1) = 0\}. \tag{2.15}$$

This establishes an inclusion $\mathsf{DMR}^G_0(\mathbf{k}) \subset \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k})$ of subgroups of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$. We then have the following result:

Corollary 2.2.5. *We have*

$$\begin{array}{rl}(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{DMR}^G_0(\mathbf{k}) \subset & \mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \\ & \cap \\ & \mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k})\end{array}$$

Proof. Thanks to Proposition 2.2.3.(ii), we have

$$\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) = (\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}). \tag{2.16}$$

On the other hand, using equality (2.15), we obtain

$$(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{DMR}^G_0(\mathbf{k}) \subset (\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}). \tag{2.17}$$

From equality (2.16) and inclusion (2.17), we obtain the inclusion

$$(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{DMR}^G_0(\mathbf{k}) \subset \mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}),$$

which is the wanted first inclusion. For the second inclusion, thanks to inclusion (2.14), we have that

$$(\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \subset (\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{Stab}_{\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k}).$$

Thanks to Proposition 2.2.3, this inclusion implies that

$$\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \subset \mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k}).$$

□

3. Construction of "Betti" coproducts

In this section, we construct a "Betti" version of the double shuffle formalism. The relevant algebras and modules are introduced in §3.1 : (i) an algebra $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ defined as the inverse limit of an algebra $\mathcal{V}_N^{\mathrm{B}}$ endowed with a suitable filtration; (ii) an algebra-module $(\widehat{\mathcal{W}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}})$ composed of a subalgebra $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ of $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ and a $\mathbf{k}$-module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ which has a $\widehat{\mathcal{V}}_N^{\mathrm{B}}$-module structure inducing a free rank one $\widehat{\mathcal{W}}_N^{\mathrm{B}}$-module structure on it. In proposition 3.1.27, we construct algebra-module isomorphisms $(\mathrm{iso}^{\mathcal{W},\iota}, \mathrm{iso}^{\mathcal{M},\iota})$ from $(\widehat{\mathcal{W}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}})$ to $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ indexed by $\iota \in \mathrm{Emb}(G)$. This gives rise to a family of algebra-module isomorphisms $\left({}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\lambda,\Psi)}, {}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}\right)$ indexed by elements $(\iota, \lambda, \Psi) \in \mathrm{Emb}(G) \times \mathbf{k}^{\times} \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. In §3.2, we show that the transport by this isomorphism of the "de Rham" pair of coproducts $(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$ is independent of the element $(\iota, \lambda, \Psi) \in \mathsf{DMR}_{\times}(\mathbf{k})$ (see Theorem 3.2.4). This is derived from the chain of inclusions of Corollary 2.2.5 and from the torsor structure of $\mathsf{DMR}_{\times}(\mathbf{k})$ over $(\mathrm{Aut}(G) \times \mathbf{k}^{\times}) \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ (see Proposition 1.4.14). The resulting pair of coproducts is denoted $(\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}, \widehat{\Delta}_N^{\mathcal{M},\mathrm{B}})$ and equips $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ with Hopf algebra and coalgebra structures respectively (see Corollary 3.2.6).

3.1. The topological algebra-module $(\widehat{\mathcal{W}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}})$.

3.1.1. *The filtered algebra $\mathcal{V}_N^{\mathrm{B}}$.* Let F_2 be the free group generated by two elements denoted X_0 and X_1. We consider the group morphism $F_2 \to \mu_N$ given by $X_0 \mapsto \zeta_N$ and $X_1 \mapsto 1$; where $\zeta_N := e^{\frac{i2\pi}{N}}$.

Lemma 3.1.1. *The group* $\ker(F_2 \to \mu_N)$ *is isomorphic to the free group of rank $N+1$ denoted F_{N+1}.*

In order to prove this, we use the following result:

Proposition 3.1.2 (Nielsen-Schreier Theorem, see [Ste, Theorem 3])**.** *Let F be a free group on a non-empty set X and let H be a subgroup of F. Let $\sigma : H\backslash F \to F$ be a section of the canonical projection $F \to H\backslash F$ such that $T := \sigma(H\backslash F)$ is stable under left prefixation. Then H is freely generated by*

$$\left\{tx(\overline{tx})^{-1} \mid (t,x) \in T \times X \text{ and } tx(\overline{tx})^{-1} \neq 1\right\},$$

where for $g \in F$, $\bar{g}$ the image of g under the composition $F \to H\backslash F \overset{\sigma}{\to} F$.

Proof of Lemma 3.1.1. We apply the Nielsen-Schreier Theorem for $X = \{X_0, X_1\}$, $F = F_2$, $H = \ker(F_2 \to \mu_N)$ and $\sigma : \ker(F_2 \to \mu_N)\backslash F_2 \simeq \mu_N \to F_2$ where the first map is the isomorphism induced by the surjective morphism $F_2 \to \mu_N$ and the second map given by $e^{i\frac{2n\pi}{N}} \mapsto X_0^n$ for $n \in [\![0, N-1]\!]$. Therefore, we have $T = \{X_0^n, n \in [\![0, N-1]\!]\}$. The theorem then states that $\ker(F_2 \to \mu_N)$ is freely generated by:

- $X_0^n X_0 (\overline{X_0^n X_0})^{-1} = X_0^{n+1}(\overline{X_0^{n+1}})^{-1} = \begin{cases} X_0^{n+1}(X_0^{n+1})^{-1} = 1 & \text{if } n \in [\![0, N-2]\!] \\ X_0^N 1^{-1} = X_0^N & \text{if } n = N-1 \end{cases}$
- $X_0^n X_1 (\overline{X_0^n X_1})^{-1} = X_0^n X_1 (X_0^n)^{-1} = X_0^n X_1 X_0^{-n}$

Finally, $\ker(F_2 \to \mu_N)$ is freely generated by the $N+1$ elements

$$\left\{X_0^N, (X_0^n X_1 X_0^{-n})_{n \in [\![0, N-1]\!]}\right\}.$$

Moreover, if we denote $\left(\widetilde{X}_0, \left(\widetilde{X}_{\zeta_N^n}\right)_{n \in [\![0,N-1]\!]}\right)$ the generators of the free group F_{N+1} of rank $N+1$, one checks that correspondence

$$\widetilde{X}_0 \mapsto X_0^N,\ \widetilde{X}_{\zeta_N^n} \mapsto X_0^n X_1 X_0^{-n} \text{ for } n \in [\![0, N-1]\!]$$

defines a free group isomorphism from F_{N+1} to $\ker(F_2 \to \mu_N)$. □

We then obtain the following short exact sequence

$$\{1\} \to F_{N+1} \to F_2 \to \mu_N \to \{1\} \tag{3.1}$$

Next, let $\sigma : \mu_N \to F_2$ be the set-theoretic section of $F_2 \to \mu_N$ given by $e^{\frac{i2n\pi}{N}} \mapsto X_0^n$ for $n \in [\![0, N-1]\!]$. Thanks to the exact sequence (3.1) we obtain a bijection

$$\Sigma : \mu_N \times F_{N+1} \to F_2, \quad (\zeta, x) \mapsto \sigma(\zeta)x; \tag{3.2}$$

where F_{N+1} is seen as $\ker(F_2 \to \mu_N) \subset F_2$ thanks to Lemma 3.1.1.
The set $\mu_N \times F_{N+1}$ is equipped with a right F_{N+1}-set structure by

$$(\zeta, x) * y := (\zeta, xy), \text{ for } (\zeta, x) \in \mu_N \times F_{N+1} \text{ and } y \in F_{N+1}.$$

The group F_2 is also equipped with a right F_{N+1}-set structure given by

$$x * y := xy, \text{ for } x \in F_2 \text{ and } y \in F_{N+1};$$

where F_{N+1} is seen as $\ker(F_2 \to \mu_N) \subset F_2$ thanks to Lemma 3.1.1. One checks that (3.2) upgrades to a right F_{N+1}-set isomorphism.
Let us consider the tensor functor

$$\mathbf{k}(-) : \{\text{right } F_{N+1}\text{-sets}\} \longrightarrow \{\text{right } \mathbf{k}F_{N+1}\text{-modules}\}$$

taking X to $\mathbf{k}X$, the set of finitely supported maps $X \to \mathbf{k}$. Applying this functor to the isomorphism of right F_{N+1}-sets (3.2), one obtains the right $\mathbf{k}F_{N+1}$-module isomorphism

$$\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}F_2, \tag{3.3}$$

where both the source and the target are equipped with the right $\mathbf{k}F_{N+1}$-module structure given by the right F_{N+1}-set structure on $\mu_N \times F_{N+1}$ and F_2 respectively.

Let us denote $\mathcal{I} := \ker(\mathbf{k}F_2 \to \mathbf{k}\mu_N)$ where $\mathbf{k}F_2 \to \mathbf{k}\mu_N$ is the $\mathbf{k}$-algebra morphism induced from the group morphism $F_2 \to \mu_N$. Then $\mathcal{I}$ is a two-sided ideal of $\mathbf{k}F_2$. In particular, $\mathcal{I}$ is a right $\mathbf{k}F_{N+1}$-module.
Let $\varepsilon : \mathbf{k}F_{N+1} \to \mathbf{k}$ be the augmentation morphism of the group algebra $\mathbf{k}F_{N+1}$. It is equipped with a right regular $\mathbf{k}F_{N+1}$-module structure.

Lemma 3.1.3.

(i) The $\mathbf{k}$*-module isomorphism* $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}F_2$ *sets up a right* $\mathbf{k}F_{N+1}$*-module isomorphism of* $\mathcal{I}$ *with* $\mathbf{k}\mu_N \otimes \ker(\varepsilon)$*.*

(ii) The ideal $\mathcal{I}$ *is linearly generated by* $\sigma(\zeta)(x-1)$ *where* $\zeta \in \mu_N$ *and* $x \in F_{N+1}$*.*

Proof.

(i) The following commutative diagram of F_{N+1}-set morphisms

$$\begin{array}{ccc} \mu_N \times F_{N+1} & \xrightarrow{\quad\Sigma\quad} & F_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow & \\ & \mu_N & \end{array}$$

induces a commutative diagram of $\mathbf{k}F_{N+1}$-module morphisms

$$\begin{array}{ccc} \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} & \xrightarrow{\quad\mathbf{k}\Sigma\quad} & \mathbf{k}F_2 \\ & {\scriptstyle \mathrm{id}\otimes\varepsilon}\searrow \quad \swarrow & \\ & \mathbf{k}\mu_N & \end{array}$$

One checks that the associated group algebra morphism of the first projection $p_1 : \mu_N \times F_{N+1} \to \mu_N$ is identified with $\mathrm{id} \otimes \varepsilon : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\mu_N$ thanks to the identification $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \simeq \mathbf{k}(\mu_N \times F_{N+1})$. Therefore, the ideal $\mathcal{I}$ is mapped by the isomorphism $\mathbf{k}\Sigma$ to the ideal $\ker(\mathrm{id} \otimes \varepsilon) = \mathbf{k}\mu_N \otimes \ker(\varepsilon)$.

(ii) Since $\varepsilon : \mathbf{k}F_{N+1} \to \mathbf{k}$ is the augmentation morphism, its kernel is generated by elements $x - 1$ with $x \in F_{N+1}$. Therefore, taking the image of the generators by $\mathbf{k}\Sigma$, we obtain generators of the ideal $\mathcal{I}$ as announced.

□

Proposition-Definition 3.1.4. *Let $\mathcal{V}_N^{\mathrm{B}}$ be the group algebra of F_2 over $\mathbf{k}$ endowed with the filtration*

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} = \mathcal{I}^m,$$

for $m \in \mathbb{N}$, where $\mathcal{I}^m$ is the m^{th}-power of the ideal $\mathcal{I}$ with the convention that $\mathcal{I}^0 = \mathcal{V}_N^{\mathrm{B}}$. The filtration $(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ is an algebra filtration.

Proof. Immediate. □

Lemma 3.1.5. *Let $m \in \mathbb{N}$. The $\mathbf{k}$-module isomorphism $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}F_2$ sets up a right $\mathbf{k}F_{N+1}$-module isomorphism of $\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$ with $\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m$, where $(\mathbf{k}F_{N+1})_0$ is the augmentation ideal of the group algebra $\mathbf{k}F_{N+1}$.*

Proof. If $m = 0$, we have $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \simeq \mathbf{k}(\mu_N \times F_{N+1}) \xrightarrow{\mathbf{k}\Sigma} \mathcal{V}_N^{\mathrm{B}} = \mathcal{F}^0\mathcal{V}_N^{\mathrm{B}}$.
Next, if $m = 1$, we have

$$\mathcal{F}^1\mathcal{V}_N^{\mathrm{B}} = \mathcal{I} \simeq \mathbf{k}\mu_N \otimes \ker(\varepsilon) = \mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0,$$

where the identification is given by Lemma 3.1.3 (i).
Now, let $m \geq 2$. Since $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1}$ is a right $\mathbf{k}F_{N+1}$-module, we have that

$$\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m = (\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0) \cdot (\mathbf{k}F_{N+1})_0^{m-1}. \tag{3.4}$$

The composition $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \simeq \mathbf{k}(\mu_N \times F_{N+1}) \xrightarrow{\mathbf{k}\Sigma} \mathcal{V}_N^{\mathrm{B}}$ is a right $\mathbf{k}F_{N+1}$-module isomorphism which, combined with the identification $\mathcal{I} \simeq \mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0$ and equality (3.4), gives us

$$\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m \simeq \mathcal{I} \cdot (\mathbf{k}F_{N+1})_0^{m-1},$$

where $(\mathbf{k}F_{N+1})_0^{m-1}$ is seen as a subset of $\mathbf{k}F_{N+1} = \mathbf{k}\ker(F_2 \to \mu_N) \subset \mathbf{k}F_2$.
It remains to show that $\mathcal{I} \cdot (\mathbf{k}F_{N+1})_0^{m-1} = \mathcal{I}^m$. First, since $(\mathbf{k}F_{N+1})_0 \subset \mathcal{I}$, we have

$\mathcal{I}\cdot(\mathbf{k}F_{N+1})_0^{m-1}\subset\mathcal{I}^m$. Conversely, thanks to Lemma 3.1.3 (ii), $\mathcal{I}^m$ is linearly generated by elements

$$\Pi((\zeta_1,x_1),\dots,(\zeta_m,x_m)) := \sigma(\zeta_1)(x_1-1)\cdots\sigma(\zeta_m)(x_m-1)$$

with $(\zeta_1,x_1),\dots,(\zeta_m,x_m)\in\mu_N\times F_{N+1}$. Moreover, we have that

$$\Pi((\zeta_1,x_1),\dots,(\zeta_m,x_m)) = \sigma(\zeta_1)\cdots\sigma(\zeta_m)\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_2)^{-1}}(x_1)-1\right)$$
$$\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_3)^{-1}}(x_2)-1\right)\cdots\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}}(x_{m-1})-1\right)\,(x_m-1).$$

Next, since F_{N+1} is a normal subgroup of F_2, we have that

$$\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_3)^{-1}}(x_2)-1\right)\cdots\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}}(x_{m-1})-1\right)\,(x_m-1)\in(\mathbf{k}F_{N+1})_0^{m-1}.$$

In addition, thanks to Lemma 3.1.3 (ii), we have

$$\sigma(\zeta_1)\cdots\sigma(\zeta_m)\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_2)^{-1}}(x_1)-1\right)\in\mathbf{k}F_2\cdot(\mathbf{k}F_{N+1})_0.$$

Since $(\mathbf{k}F_{N+1})_0\subset\mathcal{I}$, it follows that $\mathbf{k}F_2\cdot(\mathbf{k}F_{N+1})_0\subset\mathbf{k}F_2\cdot\mathcal{I}$ and since $\mathcal{I}$ is a two-sided ideal of $\mathbf{k}F_2$, we have $\mathbf{k}F_2\cdot\mathcal{I}=\mathcal{I}$. Therefore,

$$\sigma(\zeta_1)\cdots\sigma(\zeta_m)\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_2)^{-1}}(x_1)-1\right)\in\mathcal{I},$$

and then $\Pi((\zeta_1,x_1),\dots,(\zeta_m,x_m))\in\mathcal{I}\cdot(\mathbf{k}F_{N+1})_0^{m-1}$, thus proving the wanted inclusion. □

3.1.2. *The topological algebra* $\widehat{\mathcal{V}}_N^{\mathrm{B}}$. The decreasing filtration $(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ given in Proposition-Definition 3.1.4 induces an algebra morphism $\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{V}_N^{\mathrm{B}}\to\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$. One defines

Definition 3.1.6. We denote

$$\widehat{\mathcal{V}}_N^{\mathrm{B}} := \varprojlim \mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$$

the inverse limit of the system $\left(\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}, \mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{V}_N^{\mathrm{B}}\to\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}\right)$.

The algebra $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ is equipped with the filtration $\mathcal{F}^m\widehat{\mathcal{V}}_N^{\mathrm{B}} := \varprojlim \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^{\max(m,l)}\mathcal{V}_N^{\mathrm{B}}$. When equipped with the topology defined by this filtration, $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ is a complete separated topological algebra.

Recall that $\mathbf{k}F_{N+1}$ is a group algebra equipped with a filtration given by the powers of its augmentation ideal. Let us denote $\widehat{\mathbf{k}F_{N+1}}$ the completion of this group algebra with respect to this filtration.

Lemma 3.1.7.

(i) The $\mathbf{k}$*-algebra morphism* $\mathbf{k}\Sigma\circ(1\otimes -):\mathbf{k}F_{N+1}\to\mathcal{V}_N^{\mathrm{B}}$ *gives rise to a topological* $\mathbf{k}$*-algebra morphism* $\widehat{\mathbf{k}F_{N+1}}\to\widehat{\mathcal{V}}_N^{\mathrm{B}}$.

(ii) The $\mathbf{k}$*-module morphism* $\mathbf{k}\Sigma:\mathbf{k}\mu_N\otimes\mathbf{k}F_{N+1}\to\mathcal{V}_N^{\mathrm{B}}$ *gives rise to an isomorphism of topological right* $\widehat{\mathbf{k}F_{N+1}}$*-module* $\widehat{\mathbf{k}\Sigma}:\mathbf{k}\mu_N\otimes\widehat{\mathbf{k}F_{N+1}}\to\widehat{\mathcal{V}}_N^{\mathrm{B}}$.

(iii) The $\mathbf{k}$*-algebra morphism* $\widehat{\mathbf{k}F_{N+1}}\to\widehat{\mathcal{V}}_N^{\mathrm{B}}$ *is injective.*

Proof.

(i) This follows from the fact that the $\mathbf{k}$-algebra morphism $\mathbf{k}\Sigma\circ(1\otimes -):\mathbf{k}F_{N+1}\to\mathcal{V}_N^{\mathrm{B}}$ is compatible with filtrations, which follows from Lemma 3.1.5.

(ii) This follows from the fact that $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathcal{V}_N^{\mathrm{B}}$ is an isomorphism of filtered right module over $\mathbf{k}F_{N+1}$ (see (3.3)).

(iii) By (i), the topological $\mathbf{k}$-algebra morphism $\widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$ is equal to the composition $\widehat{\mathbf{k}\Sigma} \circ (1 \otimes -) : \widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$. The map $1 \otimes - : \widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ is trivially injective and $\widehat{\mathbf{k}\Sigma} : \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$ is injective by (ii). This implies that their composition is injective, implying the claim.

□

Proposition-Definition 3.1.8. *Let $\iota \in \mathrm{Emb}(G)$. There is a unique topological algebra isomorphism* $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ *given by*

$$X_0 \mapsto \exp\left(\frac{1}{N}e_0\right) g_\iota; \quad \text{and} \quad X_1 \mapsto \exp(e_1),$$

where $g_\iota = \iota^{-1}(e^{\frac{i2\pi}{N}})$.

Proof. Recall that the set $\mathrm{Mor}_{\mathbf{k}\text{-alg}}(\mathbf{k}F_2, \widehat{\mathcal{V}}_G^{\mathrm{DR}})$ is identified with $\mathrm{Mor}_{\mathrm{grp}}\Big(F_2, (\widehat{\mathcal{V}}_G^{\mathrm{DR}})^\times\Big)$. As a consequence, there is an algebra morphism $\mathcal{V}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ given by

$$X_0 \mapsto \exp\left(\frac{1}{N}e_0\right) g_\iota \quad \text{and} \quad X_1 \mapsto \exp(e_1)$$

since the images of X_0 and X_1 are invertible. Composing the $\mathbf{k}$-algebra morphism $\mathcal{V}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ with the $\mathbf{k}$-module isomorphism $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathcal{V}_N^{\mathrm{B}}$ and the inverse of the $\mathbf{k}$-algebra isomorphism $\beta : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ respectively from the left and from the right, we obtain a $\mathbf{k}$-module morphism

$$\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G. \tag{3.5}$$

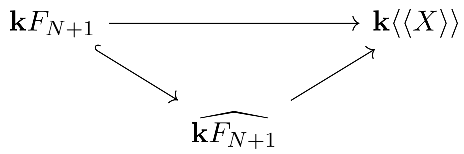
One checks that morphism (3.5) is a right module morphism over the $\mathbf{k}$-algebra morphism $\mathbf{k}F_{N+1} \to \mathbf{k}\langle\langle X\rangle\rangle$ given by

$$\widetilde{X}_0 \mapsto \exp(x_0) \text{ and } \widetilde{X}_{\zeta_N^n} \mapsto \exp\left(\frac{n}{N}x_0\right)\exp(-x_{g_\iota^n})\exp\left(-\frac{n}{N}x_0\right), \text{ for } n \in [\![0, N-1]\!].$$

In addition, $(\zeta_N^l \otimes 1)_{l\in[\![0,N-1]\!]}$ and $\big(\exp\left(\frac{l}{N}x_0\right) \otimes g_\iota^l\big)_{l\in[\![0,N-1]\!]}$ are bases of $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1}$ and $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ respectively and the morphism (3.5) induces the following bijection between the bases

$$\zeta_N^l \otimes 1 \mapsto \exp\left(\frac{l}{N}x_0\right) \otimes g_\iota^l, \text{ for } l \in [\![0, N-1]\!]. \tag{3.6}$$

Furthermore, there is a topological $\mathbf{k}$-algebra isomorphism $\widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle$ such that the following diagram

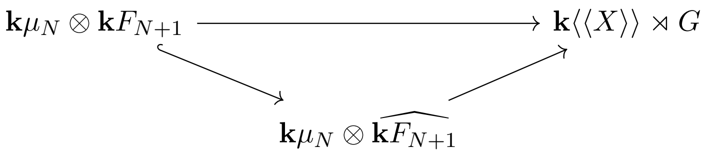


commutes, where $\mathbf{k}F_{N+1} \hookrightarrow \widehat{\mathbf{k}F_{N+1}}$ is the canonical $\mathbf{k}$-algebra morphism.
Indeed, such an isomorphism is obtained by composing the topological $\mathbf{k}$-algebra isomorphism $\widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle$ obtained from [Qui, Example A2.12] and the topological $\mathbf{k}$-algebra automorphism of $\widehat{\mathbf{k}F_{N+1}}$ given by

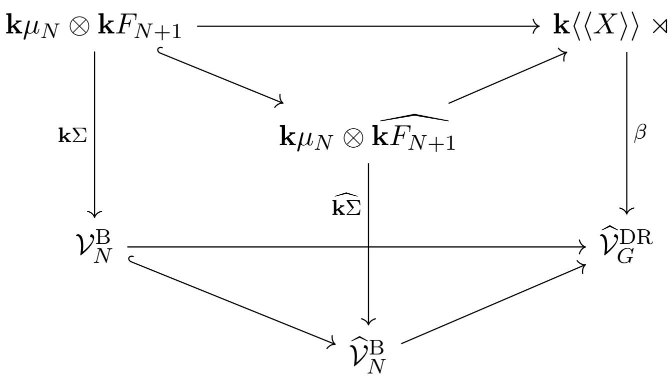
$$\widetilde{X}_0 \mapsto \widetilde{X}_0 \text{ and } \widetilde{X}_{\zeta_N^n} \mapsto \mathrm{Ad}_{\exp\left(\frac{n}{N}\log(\widetilde{X}_0)\right)}(\widetilde{X}_{\zeta_N^n}^{-1}) \text{ for } n \in [\![0, N-1]\!].$$

On the other hand, one checks that $\mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ is a free right $\widehat{\mathbf{k}F_{N+1}}$-module with basis $(\zeta_N^l \otimes 1)_{l\in[\![0,N-1]\!]}$ and recall that $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ is a free right $\mathbf{k}\langle\langle X\rangle\rangle$-module with basis $\left(\exp\left(\frac{l}{N}x_0\right) \otimes g_\iota^l\right)_{l\in[\![0,N-1]\!]}$. Therefore, there is a unique module isomorphism $\mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ over the $\mathbf{k}$-algebra isomorphism $\widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle$ which extends bijection (3.6) between bases. Therefore, the restriction to the bases of the following diagram

(3.7)



commutes, where $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ is the tensor product of the identity of $\mathbf{k}\mu_N$ with $\mathbf{k}F_{N+1} \hookrightarrow \widehat{\mathbf{k}F_{N+1}}$. This implies that the diagram commutes.
Next, by composing the $\mathbf{k}$-module isomorphism $\mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ from the left and from the right with the isomorphisms $\mathbf{k}\Sigma^{-1} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ and $\beta : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$ respectively, we obtain a $\mathbf{k}$-module isomorphism $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$.
Let us prove that this $\mathbf{k}$-module isomorphism is a $\mathbf{k}$-algebra isomorphism. It is, therefore, enough to show that it is a $\mathbf{k}$-algebra morphism. Let us consider the following prism



The left, right and middle squares commute by definition of $\widehat{\mathbf{k}\Sigma}$, $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ and $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ respectively and the upper triangle is Diagram (3.7), so is commutative. Additionally, the arrows going from the upper triangle to the lower triangle are isomorphisms. Therefore, the lower triangle is commutative. The restriction of the topological $\mathbf{k}$-module isomorphism $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ to $\mathcal{V}_N^{\mathrm{B}}$ is an algebra morphism, which by the density of $\mathcal{V}_N^{\mathrm{B}}$ in $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ implies that $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ is a topological

$\mathbf{k}$-algebra morphism and therefore a topological $\mathbf{k}$-algebra isomorphism. Finally, the commutativity of the triangle also implies that the $\mathbf{k}$-algebra isomorphism $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ is as announced. □

Proposition 3.1.9. *Let $\iota \in \mathrm{Emb}(G)$ and $\phi \in \mathrm{Aut}(G)$. We have*

$$\mathrm{iso}^{\mathcal{V},\iota\circ\phi^{-1}} = \eta_\phi^{\mathcal{V}} \circ \mathrm{iso}^{\mathcal{V},\iota},$$

with $\eta_\phi^{\mathcal{V}} \in \mathrm{Aut}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}})$ given in (1.35).

Proof. Since both sides are given as a composition of topological $\mathbf{k}$-algebra morphisms, let us the equality by checking on the family of topological generators:

$$\mathrm{iso}^{\mathcal{V},\iota\circ\phi^{-1}}(X_1) = \exp(e_1) = \eta_\phi^{\mathcal{V}} \circ \mathrm{iso}^{\mathcal{V},\iota}(X_1)$$

and

$$\begin{aligned}\mathrm{iso}^{\mathcal{V},\iota\circ\phi^{-1}}(X_0) &= \exp\left(\frac{1}{N}e_0\right) g_{\iota\circ\phi^{-1}} = \exp\left(\frac{1}{N}e_0\right)\phi(g_\iota) = \eta_\phi^{\mathcal{V}}\left(\exp\left(\frac{1}{N}e_0\right) g_\iota\right) \\ &= \eta_\phi^{\mathcal{V}} \circ \mathrm{iso}^{\mathcal{V},\iota}(X_0)\end{aligned}$$

□

3.1.3. *The filtered algebra $\mathcal{W}_N^{\mathrm{B}}$.*

Proposition-Definition 3.1.10. *Let us denote*

$$\mathcal{W}_N^{\mathrm{B}} := \mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1). \tag{3.8}$$

It is a subalgebra of $\mathcal{V}_N^{\mathrm{B}}$ endowed with the filtration

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} := \mathcal{W}_N^{\mathrm{B}} \cap \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \tag{3.9}$$

for $m \in \mathbb{N}$. The filtration $(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ is an algebra filtration.

Proof. Immediate. □

Lemma 3.1.11. *For $m \in \mathbb{N}^*$, we have*

(i) $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$. *(ii)* $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$.

Proof.

(i) Let $m \in \mathbb{N}^*$. We have

$$\begin{aligned}\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} &= \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \big(\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)\big) \\ &= \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \big(\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) \cap (\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1))\big) \\ &= \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1),\end{aligned}$$

where the second equality follows from the inclusion $\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k})$ since

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} = \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N)^m \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N) \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}), \tag{3.10}$$

where the last inclusion of (3.10) is a consequence of the fact that $\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}$ is the composition $\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N \to \mathbf{k}$ (the maps with target $\mathbf{k}$ being the augmentation morphisms). The third equality follows from

$$\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) \cap (\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)) = \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$$

which, in turn, follows from the fact that $\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) \cap (\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1))$ is the kernel of the composed map $\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}$ which is the identity on $\mathbf{k}$ and takes $\mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$ to 0. Its kernel is therefore $\mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$.

(ii) Recall from Lemma 3.1.5 that, for $m \in \mathbb{N}^*$, the $\mathbf{k}$-module isomorphism $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathcal{V}_N^{\mathrm{B}}$ induces an isomorphism

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \simeq \mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m, \tag{3.11}$$

where $(\mathbf{k}F_{N+1})_0$ is the augmentation ideal of the group algebra $\mathbf{k}F_{N+1}$. The isomorphism $\mathbf{k}\Sigma$ also induces an isomorphism

$$\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \simeq \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1}(\widetilde{X}_{\zeta_N^0} - 1).$$

Thanks to Lemma 3.11 (i), this induces the isomorphism

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \simeq \mathbf{k}\mu_N \otimes \left((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_{\zeta_N^0} - 1)\right).$$

Next, thanks to [Wei, Proposition 6.2.6], we have a $\mathbf{k}F_{N+1}$-module isomorphism $(\mathbf{k}F_{N+1})^{\oplus(N+1)} \to (\mathbf{k}F_{N+1})_0$. This isomorphism induces the following isomorphisms

$$\mathbf{k}F_{N+1} \oplus \{0\}^{\oplus N} \simeq \mathbf{k}F_{N+1}(X_{\zeta_N^0} - 1) \text{ and } (\mathbf{k}F_{N+1})_0^{m-1})^{\oplus(N+1)} \simeq (\mathbf{k}F_{N+1})_0^m,$$

where for the latter one we use the fact that $(\mathbf{k}F_{N+1})_0^m = (\mathbf{k}F_{N+1})_0^{m-1}(\mathbf{k}F_{N+1})_0$ and the fact that $(\mathbf{k}F_{N+1})_0^{m-1}$ is an ideal of $\mathbf{k}F_{N+1}$.

On the other hand, using the inclusion $(\mathbf{k}F_{N+1})_0^{m-1} \subset \mathbf{k}F_{N+1}$ and the isomorphism $\mathbf{k}F_{N+1} \oplus \{0\}^{\oplus N} \simeq \mathbf{k}F_{N+1}(X_{\zeta_N^0} - 1)$, one obtains

$$(\mathbf{k}F_{N+1})_0^{m-1})^{\oplus(N+1)} \cap \left(\mathbf{k}F_{N+1} \oplus \{0\}^{\oplus N}\right) = (\mathbf{k}F_{N+1})_0^{m-1} \oplus \{0\}^{\oplus N}.$$

Finally, one checks that the isomorphism $(\mathbf{k}F_{N+1})^{\oplus(N+1)} \to (\mathbf{k}F_{N+1})_0$ induces an isomorphism

$$(\mathbf{k}F_{N+1})_0^{m-1} \oplus \{0\}^{\oplus N} \simeq (\mathbf{k}F_{N+1})_0^{m-1}(\widetilde{X}_{\zeta_N^0} - 1)$$

and using (3.11) for m replaced by $m-1$, together with the fact that $\mathbf{k}\Sigma$ intertwines right multiplication by $X_1 - 1$ on $\mathcal{V}_N^{\mathrm{B}}$ with the tensor product of the identity on $\mathbf{k}\mu_N$ with right multiplication by $\widetilde{X}_{\zeta_N^0} - 1$ on $\mathbf{k}F_{N+1}$ implies

$$\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^{m-1}(\widetilde{X}_{\zeta_N^0} - 1) \simeq \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1),$$

thus proving the wanted result.

□

3.1.4. *The topological algebra* $\widehat{\mathcal{W}}_N^{\mathrm{B}}$. The decreasing filtration $(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ given in (3.9) induces an algebra morphism $\mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{W}_N^{\mathrm{B}} \to \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}$.

Definition 3.1.12. We denote

$$\widehat{\mathcal{W}}_N^{\mathrm{B}} := \varprojlim \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}$$

the inverse limit of the projective system $(\mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}, \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{W}_N^{\mathrm{B}} \to \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})$.

The algebra $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ equipped with filtration

$$\mathcal{F}^m\widehat{\mathcal{W}}_N^{\mathrm{B}} := \varprojlim \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^{\max(m,l)}\mathcal{W}_N^{\mathrm{B}}$$

and endowed with the topology defined by this filtration is a complete separated topological algebra.

Lemma 3.1.13. *The* $\mathbf{k}$*-algebra inclusion* $\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$ *gives rise to an injective morphism of topological* $\mathbf{k}$*-algebras* $\widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$.

Proof. This follows from the compatibility of the inclusion $\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$ with filtrations and the fact that the filtration on $\mathcal{W}_N^{\mathrm{B}}$ is induced by that of $\mathcal{V}_N^{\mathrm{B}}$. □

Proposition 3.1.14. *The topological algebra* $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ *is isomorphic to the topological subalgebra* $\mathbf{k} \oplus \widehat{\mathcal{V}}_N^{\mathrm{B}}(X_1 - 1)$ *of* $\widehat{\mathcal{V}}_N^{\mathrm{B}}$.

Proof. This will be done following this program:

Step 1: Construction of the topological $\mathbf{k}$-module $\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$.
Let us define a $\mathbf{k}$-submodule $\mathcal{W}_{N,+}^{\mathrm{B}} := \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{W}_N^{\mathrm{B}}$. It is equipped with the filtration

$$\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} := \mathcal{W}_{N,+}^{\mathrm{B}} \cap \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \text{ , for } m \in \mathbb{N}$$

induced by the inclusion $\mathcal{W}_{N,+}^{\mathrm{B}} \subset \mathcal{W}_N^{\mathrm{B}}$. Denote as follows the associated inverse limit

$$\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} := \varprojlim \mathcal{W}_{N,+}^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}}.$$

One checks that the $\mathbf{k}$-module inclusion $\mathcal{W}_{N,+}^{\mathrm{B}} \subset \mathcal{W}_N^{\mathrm{B}}$ is compatible with the filtrations, which induces a morphism of topological $\mathbf{k}$-modules $\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \to \widehat{\mathcal{W}}_N^{\mathrm{B}}$. As the filtration of $\mathcal{W}_{N,+}^{\mathrm{B}}$ is induced by that of $\mathcal{W}_N^{\mathrm{B}}$, this morphism is injective. Thanks to Lemma 3.1.13, we then have a chain of injections

$$\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{W}}_N^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{V}}_N^{\mathrm{B}}. \tag{3.12}$$

On the other hand, for any $m \in \mathbb{N}^*$, we have

$$\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} = \mathcal{W}_{N,+}^{\mathrm{B}} \cap \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \cap \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) = \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1),$$

where the second equality comes from Lemma 3.1.11 (ii). Therefore, for any $m \in \mathbb{N}$,

$$\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} = \begin{cases} \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) & \text{if } m = 0 \\ \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) & \text{otherwise} \end{cases} \tag{3.13}$$

Moreover, let us notice that $\mathcal{W}_N^{\mathrm{B}} = \mathbf{k} \oplus \mathcal{W}_{N,+}^{\mathrm{B}}$. Using (3.13) we obtain

$$\mathcal{F}^0\mathcal{W}_N^{\mathrm{B}} = \mathbf{k} \oplus \mathcal{F}^0\mathcal{W}_{N,+}^{\mathrm{B}};$$
$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}}, \text{ for } m \in \mathbb{N}^*.$$

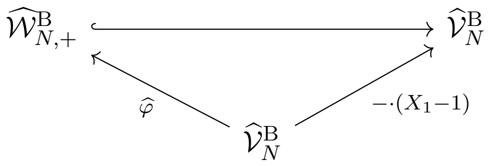
These equalities induce the following topological $\mathbf{k}$-algebra isomorphism

$$\widehat{\mathcal{W}}_N^{\mathrm{B}} = \varprojlim \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \simeq \mathbf{k} \oplus \varprojlim \mathcal{W}_{N,+}^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} = \mathbf{k} \oplus \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}. \tag{3.14}$$

where, on the right, the algebra structure is defined by the conditions that $1 \in \mathbf{k}$ is a unit and that the inclusion $\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \subset \mathbf{k} \oplus \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$ is a non-unital algebra morphism.

Step 2: The existence of a topological $\mathbf{k}$-module morphism $\widehat{\varphi} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$ such that the triangle

(3.15)



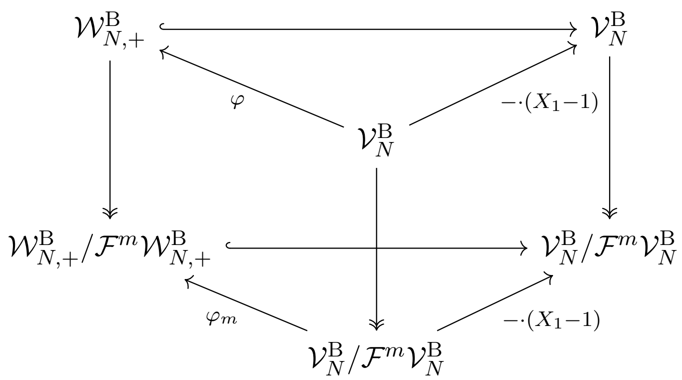
commutes. First, let us consider the $\mathbf{k}$-module morphism $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ given by $v \mapsto v(X_1 - 1)$. For any $m \in \mathbb{N}^*$, one has

$$\varphi(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}) = \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) = \mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}},$$

where the first equality follows from the definition of φ, the inclusion follows from decreasing character of $(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ and the last equality follows from (3.13). One also has

$$\varphi(\mathcal{F}^0\mathcal{V}_N^{\mathrm{B}}) = \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) = \mathcal{F}^0\mathcal{W}_{N,+}^{\mathrm{B}}.$$

This implies that the morphism $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ is compatible with filtrations. This induces a $\mathbf{k}$-module morphism $\varphi_m : \mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}}$.
In the following prism



the upper triangle commutes by definition of $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ and all the squares commute thanks to the compatibility of the maps $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$, $- \cdot (X_1 - 1) : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{V}_N^{\mathrm{B}}$ and $\mathcal{W}_{N,+}^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$ with filtrations. Therefore, thanks to the surjectivity of the projection $\mathcal{V}_N^{\mathrm{B}} \to \mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$, the lower triangle commutes. As a consequence, the morphism $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ induces a morphism of topological $\mathbf{k}$-modules $\widehat{\varphi} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$ such that Diagram (3.15) commutes. Finally, the commutativity of the latter diagram implies

$$\begin{aligned}\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_1 - 1) &= \mathrm{Im}\big(- \cdot(X_1 - 1)\big) \\ &= \mathrm{Im}\left(\widehat{\mathcal{V}}_N^{\mathrm{B}} \overset{\widehat{\varphi}}{\to} \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{V}}_N^{\mathrm{B}}\right) \subset \mathrm{Im}\left(\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{V}}_N^{\mathrm{B}}\right).\end{aligned} \tag{3.16}$$

Step 3: The existence of a topological $\mathbf{k}$-module morphism $\tilde{\phi} : \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$ such that the triangle

$$\begin{array}{ccc} \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} & \hookrightarrow & \widehat{\mathcal{V}}^{\mathrm{B}}_N \\ & {\scriptstyle \tilde{\phi}}\searrow \qquad \nearrow{\scriptstyle -\cdot(X_1-1)} & \\ & \widehat{\mathcal{V}}^{\mathrm{B}}_N & \end{array} \tag{3.17}$$

commutes. First, one notices that $\varphi : \mathcal{V}^{\mathrm{B}}_N \to \mathcal{W}^{\mathrm{B}}_{N,+}$ is a surjective $\mathbf{k}$-module morphism. It is injective thanks to the integral domain status of the algebra $\mathcal{V}^{\mathrm{B}}_N$. Therefore, the map $\varphi : \mathcal{V}^{\mathrm{B}}_N \to \mathcal{W}^{\mathrm{B}}_{N,+}$ is a $\mathbf{k}$-module isomorphism whose inverse will be denoted $\phi : \mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{V}^{\mathrm{B}}_N$. Thanks to (3.13), the $\mathbf{k}$-module isomorphism $\phi : \mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{V}^{\mathrm{B}}_N$ restricts to an isomorphism $\mathcal{F}^m\mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$, for any $m \in \mathbb{N}^*$. This induces a $\mathbf{k}$-module isomorphism $\phi_m : \mathcal{W}^{\mathrm{B}}_{N,+}/\mathcal{F}^m\mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$, for any $m \in \mathbb{N}^*$ and, via a prism similar to the one of Step 2, one checks that the following triangle

$$\begin{array}{ccc} \mathcal{W}^{\mathrm{B}}_{N,+}/\mathcal{F}^m\mathcal{W}^{\mathrm{B}}_{N,+} & \hookrightarrow & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \\ & {\scriptstyle \phi_m}\searrow \qquad \nearrow{\scriptstyle -\cdot(X_1-1)} & \\ & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N & \end{array}$$

commutes where the morphism $-\cdot(X_1-1) : \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N \to \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N$ is well-defined thanks to the inclusion $\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N(X_1-1) \subset \mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N$ being a consequence of (3.13). On the other hand, we have, for any $m \in \mathbb{N}^*$, the following triangle

$$\begin{array}{ccc} & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N & \\ & {\scriptstyle -\cdot(X_1-1)}\nearrow \qquad \nwarrow{\scriptstyle -\cdot(X_1-1)} & \\ \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N & \overset{\pi_m}{\twoheadleftarrow} & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \end{array}$$

where $\pi_m : \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \twoheadrightarrow \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$ is the morphism which associates to the class of an element modulo $\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N$, its class modulo $\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$; this is well-defined and surjective thanks to the inclusion $\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \subset \mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$. One then checks that this triangle commutes. By linking the two triangles and doing the inverse limit we obtain the following diagram

$$\begin{array}{cccc} \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} & \hookrightarrow & \widehat{\mathcal{V}}^{\mathrm{B}}_N & \\ & {\scriptstyle \widehat{\phi}}\searrow \qquad \nearrow & & \nwarrow{\scriptstyle -\cdot(X_1-1)} \\ & \varprojlim \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N & \overset{\widehat{\pi}}{\longleftarrow} & \widehat{\mathcal{V}}^{\mathrm{B}}_N \end{array}$$

where $\widehat{\pi} := \varprojlim \pi_m : \widehat{\mathcal{V}}^{\mathrm{B}}_N \to \varprojlim \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$ is obtained by degree shifting and is therefore a topological $\mathbf{k}$-module isomorphism. Let us set $\widetilde{\phi} := \widehat{\pi}^{-1} \circ \widehat{\phi} : \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$. It is a topological $\mathbf{k}$-module morphism such that Diagram (3.17) commutes. Finally, the commutativity of the latter diagram implies

$$\begin{aligned}\mathrm{Im}\left(\widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \hookrightarrow \widehat{\mathcal{V}}^{\mathrm{B}}_{N}\right) &= \mathrm{Im}\big(- \cdot (X_1 - 1) \circ \tilde{\phi}\big) \\ &\subset \mathrm{Im}\big(- \cdot (X_1 - 1)\big) = \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1).\end{aligned} \tag{3.18}$$

Finally, combining inclusions (3.16) and (3.18) we obtain

$$\widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \simeq \mathrm{Im}\left(\widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \hookrightarrow \widehat{\mathcal{V}}^{\mathrm{B}}_{N}\right) = \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1).$$

In addition, thanks to 3.14, the topological $\mathbf{k}$-algebras $\mathbf{k} \oplus \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+}$ and $\widehat{\mathcal{W}}^{\mathrm{B}}_{N}$ are isomorphic. One then obtains the isomorphism of topological $\mathbf{k}$-algebras

$$\widehat{\mathcal{W}}^{\mathrm{B}}_{N} \simeq \mathbf{k} \oplus \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1).$$

□

Proposition-Definition 3.1.15. *Let $\iota \in \mathrm{Emb}(G)$. There exists a topological algebra isomorphism* $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}^{\mathrm{B}}_{N} \to \widehat{\mathcal{W}}^{\mathrm{DR}}_{G}$ *such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{W}}^{\mathrm{B}}_{N} & \xrightarrow{\mathrm{iso}^{\mathcal{W},\iota}} & \widehat{\mathcal{W}}^{\mathrm{DR}}_{G} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{V}}^{\mathrm{B}}_{N} & \xrightarrow[\mathrm{iso}^{\mathcal{V},\iota}]{} & \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} \end{array} \tag{3.19}$$

commutes.

Proof. We have

$$\mathrm{iso}^{\mathcal{V},\iota}(X_1 - 1) = \exp(e_1) - 1 = u e_1,$$

where $u = f(e_1)$ with $f(x)$ being the invertible formal series $\frac{\exp(x)-1}{x}$. Moreover, since $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}^{\mathrm{B}}_{N} \to \widehat{\mathcal{V}}^{\mathrm{DR}}_{G}$ is a $\mathbf{k}$-algebra isomorphism, we obtain

$$\mathrm{iso}^{\mathcal{V},\iota}\left(\widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1)\right) = \mathrm{iso}^{\mathcal{V},\iota}(\widehat{\mathcal{V}}^{\mathrm{B}}_{N})\ \mathrm{iso}^{\mathcal{V},\iota}(X_1 - 1) = \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} u e_1 = \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} e_1.$$

This implies that $\mathrm{iso}^{\mathcal{V},\iota}_{|\widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1-1)} : \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1-1) \to \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} e_1$ is a surjective $\mathbf{k}$-module morphism which is trivially injective, therefore, is a $\mathbf{k}$-module isomorphism. Taking the direct sum with $\mathbf{k}$, we obtain a $\mathbf{k}$-module isomorphism

$$\mathbf{k} \oplus \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1) \to \mathbf{k} \oplus \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} e_1,$$

which is a $\mathbf{k}$-algebra isomorphism. Finally, thanks to Lemma 3.1.14, this isomorphism is the wanted $\mathbf{k}$-algebra isomorphism $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}^{\mathrm{B}}_{N} \to \widehat{\mathcal{W}}^{\mathrm{DR}}_{G}$. □

Corollary 3.1.16. *Let $\iota \in \mathrm{Emb}(G)$ and $\phi \in \mathrm{Aut}(G)$. We have*

$$\mathrm{iso}^{\mathcal{W},\iota\circ\phi^{-1}} = \eta^{\mathcal{W}}_{\phi} \circ \mathrm{iso}^{\mathcal{W},\iota},$$

with $\eta^{\mathcal{W}}_{\phi} \in \mathrm{Aut}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{W}}^{\mathrm{DR}}_{G})$ given in Lemma 1.4.8.(i).

Proof. The statement follows from Proposition 3.1.9 thanks to the commutativity of diagrams (3.19) and (1.37). □

3.1.5. *The filtered module $\mathcal{M}_N^{\mathrm{B}}$.*

Proposition-Definition 3.1.17. *The quotient* $\mathbf{k}$*-module*

$$\mathcal{M}_N^{\mathrm{B}} := \mathcal{V}_N^{\mathrm{B}} \Big/ \mathcal{V}_N^{\mathrm{B}}(X_0 - 1) \tag{3.20}$$

is a $\mathcal{V}_N^{\mathrm{B}}$-module. Moreover, if we denote 1_{B} the class of $1 \in \mathcal{V}_N^{\mathrm{B}}$ in $\mathcal{M}_N^{\mathrm{B}}$, then the canonical projection

$$- \cdot 1_{\mathrm{B}} : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$$

is a surjective $\mathcal{V}_N^{\mathrm{B}}$-module morphism and its restriction to $\mathcal{W}_N^{\mathrm{B}}$ is a $\mathcal{W}_N^{\mathrm{B}}$-module isomorphism.

Proof. This follows from the direct sum decomposition

$$\mathcal{V}_N^{\mathrm{B}} = \mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \oplus \mathcal{V}_N^{\mathrm{B}}(X_0 - 1) = \mathcal{W}_N^{\mathrm{B}} \oplus \mathcal{V}_N^{\mathrm{B}}(X_0 - 1)$$

given by [Wei, Proposition 6.2.6]. □

Remark. The statement implies that $(- \cdot 1_{\mathrm{B}})_{|\mathcal{W}_N^{\mathrm{B}}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ is a $\mathcal{W}_N^{\mathrm{B}}$-module isomorphism, therefore $\mathcal{M}_N^{\mathrm{B}}$ is a free $\mathcal{W}_N^{\mathrm{B}}$-module of rank 1.

Proposition-Definition 3.1.18. *The* $\mathbf{k}$*-module $\mathcal{M}_N^{\mathrm{B}}$ is endowed with the decreasing* $\mathbf{k}$*-module filtration given by*

$$\mathcal{F}^m \mathcal{M}_N^{\mathrm{B}} := \mathcal{F}^m \mathcal{W}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} \quad \text{for } m \in \mathbb{N}. \tag{3.21}$$

Moreover, the pair $\left(\mathcal{M}_N^{\mathrm{B}}, \big(\mathcal{F}^m \mathcal{M}_N^{\mathrm{B}}\big)_{m\in\mathbb{N}}\right)$ is a filtered module over the filtered algebra $\left(\mathcal{W}_N^{\mathrm{B}}, \big(\mathcal{F}^m \mathcal{W}_N^{\mathrm{B}}\big)_{m\in\mathbb{N}}\right)$.

Proof. Immediate. □

Lemma 3.1.19.

(i) For any $m \in \mathbb{N}$, the $\mathbf{k}$*-module isomorphism $- \cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ induces a* $\mathbf{k}$*-modules isomorphism $\mathcal{F}^m \mathcal{W}_N^{\mathrm{B}} \to \mathcal{F}^m \mathcal{M}_N^{\mathrm{B}}$.*

(ii) For any $m \in \mathbb{N}$, we have $\mathcal{F}^m \mathcal{M}_N^{\mathrm{B}} = \mathcal{F}^m \mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}}$.

Proof.

(i) By definition of $\mathcal{F}^m \mathcal{M}_N^{\mathrm{B}}$, the isomorphism $- \cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ restricts to a surjective $\mathbf{k}$-module morphism $\mathcal{F}^m \mathcal{W}_N^{\mathrm{B}} \to \mathcal{F}^m \mathcal{M}_N^{\mathrm{B}}$. In addition, since $- \cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ is injective, so is the restriction $\mathcal{F}^m \mathcal{W}_N^{\mathrm{B}} \to \mathcal{F}^m \mathcal{M}_N^{\mathrm{B}}$.

(ii) First, if $m = 0$, the equality follows from Proposition-Definition 3.1.17. From now on, let $m \in \mathbb{N}^*$. Since $\mathcal{F}^m \mathcal{W}_N^{\mathrm{B}} \subset \mathcal{F}^m \mathcal{V}_N^{\mathrm{B}}$, we have that

$$\mathcal{F}^m \mathcal{M}_N^{\mathrm{B}} \subset \mathcal{F}^m \mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}}.$$

Conversely, let us prove the inclusion $\mathcal{F}^m \mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} \subset \mathcal{F}^m \mathcal{M}_N^{\mathrm{B}}$. This inclusion is equivalent to

$$\mathcal{F}^m \mathcal{V}_N^{\mathrm{B}} \subset \mathcal{F}^m \mathcal{W}_N^{\mathrm{B}} + \mathcal{V}_N^{\mathrm{B}}(X_0 - 1).$$

Since $\mathcal{F}^m \mathcal{V}_N^{\mathrm{B}} = \mathcal{I}^m = \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N)^m$ and by Lemma 3.1.11 (i), this inclusion is also equivalent to

$$\mathcal{I}^m \subset \big(\mathcal{I}^m \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)\big) + \mathcal{V}_N^{\mathrm{B}}(X_0 - 1). \tag{3.22}$$

We have

$$\mathcal{I} = \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N) \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) = \mathcal{V}_N^{\mathrm{B}}(X_0 - 1) + \mathcal{V}_N^{\mathrm{B}}(X_1 - 1),$$

with $\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k})$ being the augmentation ideal of the group algebra $\mathcal{V}_N^{\mathrm{B}} = \mathbf{k}F_2$ and the last equality being a consequence of [Wei, Proposition 6.2.6].

This implies

$$\begin{aligned} \mathcal{I}^m = \mathcal{I}^{m-1}\mathcal{I} &\subset \mathcal{I}^{m-1}\left(\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) + \mathcal{V}_N^{\mathrm{B}}(X_0 - 1)\right) \\ &\subset \mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) + \mathcal{V}_N^{\mathrm{B}}(X_0 - 1). \end{aligned} \tag{3.23}$$

Moreover, $\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N)$ since $X_1 - 1 \mapsto 0$ through the map $\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N$. This implies

$$\mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{I}^{m-1}\mathcal{I} = \mathcal{I}^m. \tag{3.24}$$

On the other hand, we have

$$\mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{V}_N^{\mathrm{B}}(X_1 - 1). \tag{3.25}$$

From (3.24) and (3.25) we obtain

$$\mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \left(\mathcal{I}^m \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)\right). \tag{3.26}$$

Finally, from (3.23) and (3.26), we obtain

$$\mathcal{I}^m \subset \left(\mathcal{I}^m \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)\right) + \mathcal{V}_N^{\mathrm{B}}(X_0 - 1),$$

which is the wanted inclusion.

□

3.1.6. *The topological module* $\widehat{\mathcal{M}}_N^{\mathrm{B}}$. The decreasing filtration $(\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ given in (3.21) induces a $\mathbf{k}$-module morphism $\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{M}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}$.

Definition 3.1.20. We denote

$$\widehat{\mathcal{M}}_N^{\mathrm{B}} := \varprojlim \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}.$$

the limit of the projective system $(\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}, \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{M}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}})$.

The $\mathbf{k}$-module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ is a $\widehat{\mathcal{V}}_N^{\mathrm{B}}$-module equipped with the filtration

$$\mathcal{F}^m\widehat{\mathcal{M}}_N^{\mathrm{B}} := \varprojlim \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^{\max(m,l)}\mathcal{M}_N^{\mathrm{B}}, \text{ for } m \in \mathbb{N}.$$

When equipped with the topology defined by this filtration, $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ is a complete separated topological $\mathbf{k}$-module.

Lemma 3.1.21.

(i) *The surjective* $\mathbf{k}$*-module morphism* $- \cdot 1_{\mathrm{B}} : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ *induces a topological surjective* $\mathbf{k}$*-module morphism* $\widehat{- \cdot 1_{\mathrm{B}}} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$.

(ii) *The* $\mathbf{k}$*-module isomorphism* $- \cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ *induces a topological* $\mathbf{k}$*-module isomorphism* $\widehat{- \cdot 1_{\mathrm{B}}} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$.

Proof.

(i) By definition of $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$, this follows from Lemma 3.1.19 (ii).

(ii) By definition of $\widehat{\mathcal{M}}_N^{\mathrm{B}}$, this follows from Lemma 3.1.19 (i).

□

Corollary 3.1.22.

(i) The pair $(\widehat{\mathcal{V}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}})$ *is an object in the category* $\mathbf{k}\text{-alg-mod}_{\mathrm{top}}$.

(ii) The pair $(\widehat{\mathcal{W}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}})$ *is an object in the category* $\mathbf{k}\text{-alg-mod}_{\mathrm{top}}$. *Moreover,* $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ *is a free* $\widehat{\mathcal{W}}_N^{\mathrm{B}}$*-module of rank* 1.

Proof. It immediately follows from Lemma 3.1.21. □

Proposition 3.1.23. *The topological* $\mathbf{k}$*-module morphism* $\widehat{-\cdot 1_{\mathrm{B}}} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$ *induces an isomorphism* $\widehat{\mathcal{V}}_N^{\mathrm{B}}/\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_0-1) \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$ *of topological* $\mathbf{k}$*-modules.*

In order to prove this, we will need the following Lemma:

Lemma 3.1.24. *Let* V *be a* $\mathbf{k}$*-module and* u *be an endomorphism of* V*. Let* $f : V^N \to V^N$ *be the endomorphism given by*

$$(v_0, \dots, v_{N-1}) \mapsto (u(v_{N-1}) - v_0, v_0 - v_1, v_1 - v_2, \dots, v_{N-2} - v_{N-1}).$$

Then we have an isomorphism

$$\mathrm{coker}(f) \simeq \mathrm{coker}(u - \mathrm{id}).$$

Proof of Lemma 3.1.24. Let us consider the $\mathbf{k}$-module morphism sum : $V^N \to V$ given by $(v_0, \dots, v_{N-1}) \mapsto v_0 + \cdots + v_{N-1}$. This morphism sends $\mathrm{Im}(f)$ to $\mathrm{Im}(u - \mathrm{id})$. Therefore, there is a unique $\mathbf{k}$-module morphism $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u - \mathrm{id})$ such that the diagram

$$\begin{array}{ccc} V^N & \xrightarrow{\quad\text{sum}\quad} & V \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ V^N/\mathrm{Im}(f) & \longrightarrow & V/\mathrm{Im}(u-\mathrm{id}) \end{array}$$

commutes. Let us show that the morphism $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u - \mathrm{id})$ is an isomorphism. First, the surjectivity of the morphism sum : $V^N \to V$ implies that the morphism $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u-\mathrm{id})$ is surjective as well. Second, let $(w_0, \dots, w_{N-1}) \in V^N$ such that there exists an element $v \in V$ such that $w_0 + \cdots w_{N-1} = u(v) - v$. The element $(v_0, \dots, v_{N-1}) \in V^N$ given by

$$v_{N-1} = v, v_{N-2} = w_{N-1} + v, v_{N-3} = w_{N-2} + w_{N-1} + v, \dots, v_0 = w_1 + \cdots w_{N-1} + v$$

is such that

$$(w_0, \dots, w_{N-1}) = (u(v_{N-1}) - v_0, v_0 - v_1, \dots, v_{N-2} - v_{N-1}) \in \mathrm{Im}(f).$$

Thus proving the injectivity of $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u - \mathrm{id})$. □

Proof of Proposition 3.1.23. The proof consists of the following steps:

Step 1: Construction of the $\mathbf{k}$-module isomorphism $P_0 : (\mathbf{k}F_{N+1})^N \to \mathcal{V}_N^{\mathrm{B}}$.
As in (3.3), one defines the $\mathbf{k}$-module isomorphism $\mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N \to \mathcal{V}_N^{\mathrm{B}}$ such that for $(x, \zeta) \in F_{N+1} \times \mu_N$

$$x \otimes \zeta \mapsto x\,\sigma(\zeta), \tag{3.27}$$

where F_{N+1} is seen as $\ker(F_2 \to \mu_N) \subset F_2$ thanks to Lemma 3.1.1. Moreover, one checks there is a $\mathbf{k}$-module isomorphism $(\mathbf{k}F_{N+1})^N \to \mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N$ given by

$$(v_0, \dots, v_{N-1}) \mapsto \sum_{i=0}^{N-1} v_i \otimes \zeta_N^i. \tag{3.28}$$

Therefore, the composition $P_0 : (\mathbf{k}F_{N+1})^N \to \mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N \to \mathcal{V}_N^{\mathrm{B}}$ is a $\mathbf{k}$-module isomorphism and is given by

$$(v_0, \dots, v_{N-1}) \mapsto v_0 + v_1 X_0 + \cdots + v_{N-1} X_0^{N-1}.$$

Step 2: Identification of $\mathcal{M}_N^{\mathrm{B}}$.
One checks that the endomorphism $f : (\mathbf{k}F_{N+1})^N \to (\mathbf{k}F_{N+1})^N$ given by

$$(v_0, \dots, v_{N-1}) \mapsto (v_{N-1}\widetilde{X}_0 - v_0, v_0 - v_1, v_1 - v_2, \dots, v_{N-2} - v_{N-1}), \tag{3.29}$$

is such that the following diagram

$$\begin{array}{ccc} (\mathbf{k}F_{N+1})^N & \xrightarrow{\quad f \quad} & (\mathbf{k}F_{N+1})^N \\ {\scriptstyle P_0}\downarrow & & \downarrow{\scriptstyle P_0} \\ \mathcal{V}_N^{\mathrm{B}} & \xrightarrow[-\cdot(X_0-1)]{} & \mathcal{V}_N^{\mathrm{B}} \end{array} \tag{3.30}$$

commutes. This induces a $\mathbf{k}$-module isomorphism $\mathrm{coker}(f) \simeq \mathrm{coker}(- \cdot (X_0 - 1))$.
On the other hand, by applying Lemma 3.1.24 with $V = \mathbf{k}F_{N+1}$ and $u = - \cdot \widetilde{X}_0$, we obtain an isomomorphism $\mathrm{coker}(f) \simeq \mathrm{coker}(u - \mathrm{id})$. It then follows that

$$\begin{aligned} \mathcal{M}_N^{\mathrm{B}} = \mathcal{V}_N^{\mathrm{B}} / \mathcal{V}_N^{\mathrm{B}}(X_0 - 1) = \mathrm{coker}(- \cdot (X_0 - 1)) \\ \simeq \mathrm{coker}(f) \simeq \mathrm{coker}(u - \mathrm{id}) = \mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1). \end{aligned}$$

Step 3: Compatibility of the isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) \to \mathcal{M}_N^{\mathrm{B}}$ with filtrations. Let us show that for any $m \in \mathbb{N}$, we have

$$(\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) \Big) \simeq \mathcal{F}^m \mathcal{M}_N^{\mathrm{B}}.$$

If $m = 0$, this has been proved in Step 2. From now on, let us assume that $m \in \mathbb{N}^*$. The isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) \to \mathcal{M}_N^{\mathrm{B}}$ fits in the following commutative diagram

$$\begin{array}{ccc} \mathbf{k}F_{N+1} & \hookrightarrow & \mathcal{V}_N^{\mathrm{B}} \\ \downarrow & & \downarrow \\ \mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) & \xrightarrow{\ \simeq\ } & \mathcal{M}_N^{\mathrm{B}} \end{array} \tag{3.31}$$

where $\mathbf{k}F_{N+1} \to \mathcal{V}_N^{\mathrm{B}}$ is the group algebra morphism induced by the group morphism $F_{N+1} \simeq \ker(F_2 \to \mu_N) \subset F_2$ obtained in Lemma 3.1.1. This group algebra morphism induces the injection

$$(\mathbf{k}F_{N+1})_0^m \hookrightarrow \mathcal{F}^m \mathcal{V}_N^{\mathrm{B}}.$$

Then, thanks to the commutativity of Diagram (3.31), the $\mathbf{k}$-module isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0-1) \to \mathcal{M}_N^{\mathrm{B}}$ induces an injection

$$(\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0-1)\Big) \hookrightarrow \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} = \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}},$$

where the equality comes from Lemma 3.1.19.(ii). This implies that

$$\mathrm{Im}\left((\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0-1)\Big) \to \mathcal{M}_N^{\mathrm{B}}\right) \subset \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}.$$

Conversely, let us show the opposite inclusion. Thanks to Lemma 3.1.11(ii), we have

$$\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} = \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1-1) \cdot 1_{\mathrm{B}}.$$

Moreover, we have by definition that $\mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}} = (\mathcal{F}^1\mathcal{V}_N^{\mathrm{B}})^{m-1}$. This implies, thanks to Lemma 3.1.3 (ii) that $\mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1-1) \cdot 1_{\mathrm{B}}$ is linearly generated by elements

$$\sigma(\zeta_1)(x_1-1)\cdots\sigma(\zeta_{m-1})(x_{m-1}-1)(X_1-1) \cdot 1_{\mathrm{B}}$$

with $(\zeta_1, x_1), \ldots, (\zeta_{m-1}, x_{m-1}) \in \mu_N \times F_{N+1}$. Additionally, we have that

$$\begin{aligned}
&\sigma(\zeta_1)(x_1-1)\cdots\sigma(\zeta_{m-1})(x_{m-1}-1)(X_1-1) \cdot 1_{\mathrm{B}} = \\
&\big(\mathrm{Ad}_{\sigma(\zeta_1)}(x_1)-1\big)\cdots\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(x_{m-1})-1\big)\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(X_1)-1\big) \\
&\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1}) \cdot 1_{\mathrm{B}} = \\
&\big(\mathrm{Ad}_{\sigma(\zeta_1)}(x_1)-1\big)\cdots\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(x_{m-1})-1\big)\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(X_1)-1\big) \cdot 1_{\mathrm{B}}
\end{aligned}$$

which belongs to the image of $(\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0-1)\Big)$ by the isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0-1) \to \mathcal{M}_N^{\mathrm{B}}$ as

$$\big(\mathrm{Ad}_{\sigma(\zeta_1)}(x_1)-1\big)\cdots\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(x_{m-1})-1\big)\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(X_1)-1\big),$$

seen as an element of $\mathbf{k}F_{N+1}$, belongs to $(\mathbf{k}F_{N+1})_0^m$. This implies that

$$\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} \subset \mathrm{Im}\left((\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0-1)\Big) \to \mathcal{M}_N^{\mathrm{B}}\right).$$

Therefore, one has equality

$$\mathrm{Im}\left((\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0-1)\Big) \to \mathcal{M}_N^{\mathrm{B}}\right) = \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}},$$

which establishes the wanted isomorphism.

Step 4: Identification of $\widehat{\mathcal{M}}_N^{\mathrm{B}}$.

Thanks to Step 3, one has for any $m \in \mathbb{N}$

$$\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} \simeq \mathbf{k}F_{N+1} \Big/ \Big(\mathbf{k}F_{N+1}(\widetilde{X}_0-1) + (\mathbf{k}F_{N+1})_0^m\Big)$$

and, on the other hand, for any $m \in \mathbb{N}^*$,

$$\mathbf{k}F_{N+1} \Big/ \Big(\mathbf{k}F_{N+1}(\widetilde{X}_0-1) + (\mathbf{k}F_{N+1})_0^m\Big) \simeq \mathrm{coker}\left(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m\right),$$

where the morphism

$$\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \tag{3.32}$$

is induced by the endomorphism $- \cdot (\widetilde{X}_0-1)$ of $\mathbf{k}F_{N+1}$. Therefore,

$$\widehat{\mathcal{M}}_N^{\mathrm{B}} \simeq \varprojlim \mathrm{coker}\left(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m\right).$$

Step 5: Identification of $\widehat{\mathcal{V}}^{\mathrm{B}}_N/\widehat{\mathcal{V}}^{\mathrm{B}}_N(X_0-1)$.
As in Lemma 3.1.5, one proves that the $\mathbf{k}$-module isomorphism $\mathbf{k}F_{N+1}\otimes\mathbf{k}\mu_N\to\mathcal{V}^{\mathrm{B}}_N$ given in (3.27) allows us to identify $(\mathbf{k}F_{N+1})^m_0\otimes\mathbf{k}\mu_N$ with $\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N$ for any $m\in\mathbb{N}$. Recall the isomorphism $(\mathbf{k}F_{N+1})^N\to\mathbf{k}F_{N+1}\otimes\mathbf{k}\mu_N$ given in (3.28). One checks that it is compatible with the filtration of $(\mathbf{k}F_{N+1})^N$ given for any $m\in\mathbb{N}$ by $\prod_{i=1}^N(\mathbf{k}F_{N+1})^m_0$. Therefore the isomorphism $P_0:(\mathbf{k}F_{N+1})^N\to\mathbf{k}F_{N+1}\otimes\mathbf{k}\mu_N\to\mathcal{V}^{\mathrm{B}}_N$ of Step 1 is compatible with filtrations. Therefore, it extends to a topological $\mathbf{k}$-module isomorphism

$$\widehat{P}_0:(\widehat{\mathbf{k}F_{N+1}})^N\to\widehat{\mathcal{V}}^{\mathrm{B}}_N.$$

On the other hand, the endomorphism $f:(\mathbf{k}F_{N+1})^N\to(\mathbf{k}F_{N+1})^N$ given in (3.29) is compatible with filtrations and then extends to a topological endomorphism $\hat{f}:(\widehat{\mathbf{k}F_{N+1}})^N\to(\widehat{\mathbf{k}F_{N+1}})^N$ and, thanks to Diagram (3.30), it is such that the following diagram

$$\begin{array}{ccc}(\widehat{\mathbf{k}F_{N+1}})^N & \xrightarrow{\hat{f}} & (\widehat{\mathbf{k}F_{N+1}})^N\\ \simeq\downarrow & & \downarrow\simeq\\ \widehat{\mathcal{V}}^{\mathrm{B}}_N & \xrightarrow[-\cdot(X_0-1)]{} & \widehat{\mathcal{V}}^{\mathrm{B}}_N\end{array}$$

commutes. This induces a $\mathbf{k}$-module isomorphism $\mathrm{coker}(\hat{f})\simeq\mathrm{coker}(-\cdot(X_0-1))$. Similarly to Step 1, by applying Lemma 3.1.24 with $V=\widehat{\mathbf{k}F_{N+1}}$ and $u=-\cdot\widetilde{X}_0$, we obtain an isomomorphism $\mathrm{coker}(\hat{f})\simeq\mathrm{coker}(u-\mathrm{id})$. It then follows that

$$\begin{aligned}\widehat{\mathcal{V}}^{\mathrm{B}}_N/\widehat{\mathcal{V}}^{\mathrm{B}}_N(X_0-1)&=\mathrm{coker}(-\cdot(X_0-1))\\ &\simeq\mathrm{coker}(\hat{f})\simeq\mathrm{coker}(u-\mathrm{id})=\widehat{\mathbf{k}F_{N+1}}/\widehat{\mathbf{k}F_{N+1}}(\widetilde{X}_0-1).\end{aligned}$$

On the other hand, we have

$$\widehat{\mathbf{k}F_{N+1}}/\widehat{\mathbf{k}F_{N+1}}(\widetilde{X}_0-1)\simeq\mathrm{coker}\left(\varprojlim\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^{m-1}_0\to\varprojlim\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^m_0\right),$$

where the morphism $\varprojlim\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^{m-1}_0\to\varprojlim\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^m_0$ is induced by the morphism $\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^{m-1}_0\to\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^m_0$ given in (3.32).
Step 6: Cokernel of limits and limit of cokernels coincide.
For any $m\in\mathbb{N}^*$, the morphism

$$\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^{m-1}_0\to\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})^m_0 \tag{3.33}$$

induced by the endomorphism $-\cdot(\widetilde{X}_0-1)$ of $\mathbf{k}F_{N+1}$ is injective. Indeed, let $x\in\mathbf{k}F_{N+1}$ such that $x(\widetilde{X}_0-1)\in(\mathbf{k}F_{N+1})^m_0$. Let l to be the smallest integer such that $x\in(\mathbf{k}F_{N+1})^l_0$. Let us show that $l\geq m-1$. Otherwise, since $x\in(\mathbf{k}F_{N+1})^l_0$, we have that $[x]\in\mathrm{gr}_l(\mathbf{k}F_{N+1})$. Moreover, we have that $[\widetilde{X}_0-1]\in\mathrm{gr}_1(\mathbf{k}F_{N+1})$. Therefore, $[x(\widetilde{X}_0-1)]\in\mathrm{gr}_{l+1}(\mathbf{k}F_{N+1})$. Since, by assumption, $l+1\leq m-1$, the condition $x(\widetilde{X}_0-1)\in(\mathbf{k}F_{N+1})^m_0$ implies that $[x(\widetilde{X}_0-1)]=0$. Finally, since $\mathrm{gr}_{l+1}(\mathbf{k}F_{N+1})$ is an integral domain we would obtain that $[x]=0$, contradicting the minimality of l. Therefore, $l\geq m-1$ and the morphism (3.33) is injective.
In addition, the image of morphism (3.33) is the same as the image of the morphism

$\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m$ induced by the endomorphism $- \cdot (\widetilde{X}_0 - 1)$ of $\mathbf{k}F_{N+1}$. We then have the short exact sequence

$$\{0\} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m$$
$$\to \operatorname{coker}\big((\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m)\big) \to \{0\}$$

which, by applying the inverse limit functor, gives us

$$\{0\} \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \to$$
$$\varprojlim \operatorname{coker}\big((\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m)\big) \to \varprojlim{}^1 \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1},$$

where $\lim_{\longleftarrow}{}^1$ is the functor given in [BK72, §IX.2.1].
Since the transition maps of the inverse system $(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1})_{m\in\mathbb{N}^*}$ are surjective, this implies that $\varprojlim{}^1 \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} = 0$ (see, for example, [BK72, Proposition IX.2.4]). As a consequence,

$$\varprojlim \operatorname{coker}\big((\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m)\big) \simeq$$
$$\operatorname{coker}\big(\varprojlim(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m)\big).$$

Thanks to Step 4 and Step 5, this proves the wanted result.

□

Proposition-Definition 3.1.25. *Let $\iota \in \mathrm{Emb}(G)$. There exists an unique topological $\mathbf{k}$-module isomorphism $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{V}}_N^{\mathrm{B}} & \xrightarrow{\mathrm{iso}^{\mathcal{V},\iota}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{-\cdot 1_{\mathrm{B}}}}\downarrow & & \downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\ \widehat{\mathcal{M}}_N^{\mathrm{B}} & \xrightarrow[\mathrm{iso}^{\mathcal{M},\iota}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{3.34}$$

commutes.

Proof. Let us construct a topological module morphism $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ over the topological algebra morphism $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$. We consider the composition

$$\widehat{\mathcal{V}}_N^{\mathrm{B}} \xrightarrow{\mathrm{iso}^{\mathcal{V},\iota}} \widehat{\mathcal{V}}_G^{\mathrm{DR}} \xrightarrow{\widehat{-\cdot 1_{\mathrm{DR}}}} \widehat{\mathcal{M}}_G^{\mathrm{DR}}. \tag{3.35}$$

This composition sends the $\mathbf{k}$-submodule $\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_0 - 1)$ to 0. Indeed, this comes from the fact that (3.35) is a module morphism over the algebra morphism $\mathrm{iso}^{\mathcal{V},\iota}$ and the following computation

$$\mathrm{iso}^{\mathcal{V},\iota}(X_0 - 1) = g_\iota \exp\left(\tfrac{1}{N} e_0\right) - 1 = g_\iota \left(\exp\left(\tfrac{1}{N} e_0\right) - 1\right) + (g_\iota - 1) \in \widehat{\mathcal{V}}^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}^{\mathrm{DR}}(g_\iota - 1)$$

Therefore, thanks to Proposition 3.1.23, the composition (3.35) factorises into a $\mathbf{k}$-module morphism $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ which is a module morphism over the algebra morphism $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$.
Next, let us show that $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is an isomorphism. Recall from Proposition 3.1.15 that $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is an algebra submorphism of $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$.

As a result, $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a module morphism over the algebra isomorphism $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. In addition, $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ are both free rank 1 modules over $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ respectively and $\mathrm{iso}^{\mathcal{M},\iota}$ sends 1_{B} to 1_{DR} and therefore a basis of the source to a basis of the target. Thus $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a module isomorphism over $\mathrm{iso}^{\mathcal{W},\iota}$ and then a $\mathbf{k}$-module isomorphism. □

Remark 3.1.26. Let us notice that we have the following equality of $\mathbf{k}$-submodules of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$:

$$\widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota \exp(e_0) - 1) = \widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota - 1).$$

Indeed, since we have that

$$g_\iota \exp(e_0) - 1 = g_\iota(\exp(e_0) - 1) + (g_\iota - 1),$$

this gives us the inclusion $\widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota \exp(e_0) - 1) \subset \widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota - 1)$. Conversely, the inclusion $\widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota - 1) \subset \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota \exp(e_0) - 1)$ follows from

$$e_0 = \frac{e_0}{\exp(N e_0) - 1}\big(1 + g_\iota \exp(e_0) + \cdots + g_\iota^{N-1} \exp((N-1)e_0)\big)(g_\iota \exp(e_0) - 1)$$

and from

$$\begin{aligned} g_\iota - 1 =& \exp(-e_0)(g_\iota \exp(e_0) - 1) + (\exp(-e_0) - 1) \\ =& \exp(-e_0)(g_\iota \exp(e_0) - 1) + \\ & \frac{\exp(-e_0) - 1}{\exp(N e_0) - 1}\big(1 + g_\iota \exp(e_0) + \cdots + g_\iota^{N-1} \exp((N-1)e_0)\big)(g_\iota \exp(e_0) - 1) \\ =& \Big(\exp(-e_0) + \tfrac{\exp(-e_0) - 1}{\exp(N e_0) - 1}\big(1 + g_\iota \exp(e_0) + \cdots + g_\iota^{N-1} \exp((N-1)e_0)\big)\Big)(g_\iota \exp(e_0) - 1). \end{aligned}$$

Proposition 3.1.27. *For any $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the following pairs are isomorphisms in the category $\mathbf{k}$-alg-mod$_{\mathrm{top}}$:*

(i) $\big(\mathrm{iso}^{\mathcal{V},\iota}, \mathrm{iso}^{\mathcal{M},\iota}\big) : (\widehat{\mathcal{V}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}}) \to (\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$.

(ii) $\big(\mathrm{iso}^{\mathcal{W},\iota}, \mathrm{iso}^{\mathcal{M},\iota}\big) : (\widehat{\mathcal{W}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}}) \to (\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$.

Proof.

(i) The fact that $\mathrm{iso}^{\mathcal{V},\iota}$ (resp. $\mathrm{iso}^{\mathcal{M},\iota}$) is a $\mathbf{k}$-algebra (resp $\mathbf{k}$-module) isomorphism follows from Proposition-Definition 3.1.8 (resp. Proposition-Definition 3.1.25). Let $(a, m) \in \widehat{\mathcal{V}}_N^{\mathrm{B}} \times \widehat{\mathcal{M}}_N^{\mathrm{B}}$. There exists $v \in \widehat{\mathcal{V}}_N^{\mathrm{B}}$ such that $m = v \cdot 1_{\mathrm{B}}$. We have

$$\begin{aligned} \mathrm{iso}^{\mathcal{M},\iota}(am) =& \mathrm{iso}^{\mathcal{M},\iota}(av \cdot 1_{\mathrm{B}}) = \mathrm{iso}^{\mathcal{V},\iota}(av) \cdot 1_{\mathrm{DR}} = \mathrm{iso}^{\mathcal{V},\iota}(a)\, \mathrm{iso}^{\mathcal{V},\iota}(v) \cdot 1_{\mathrm{DR}} \\ =& \mathrm{iso}^{\mathcal{V},\iota}(a)\, \mathrm{iso}^{\mathcal{M},\iota}(v \cdot 1_{\mathrm{B}}) = \mathrm{iso}^{\mathcal{V},\iota}(a)\, \mathrm{iso}^{\mathcal{M},\iota}(m), \end{aligned}$$

where the second and fourth equalities come from Proposition-Definition 3.1.25.

(ii) The fact that $\mathrm{iso}^{\mathcal{W},\iota}$ (resp. $\mathrm{iso}^{\mathcal{M},\iota}$) is a $\mathbf{k}$-algebra (resp $\mathbf{k}$-module) isomorphism follows from Proposition-Definition 3.1.15 (resp. Proposition-Definition 3.1.25). One proves, for any $(w, m) \in \widehat{\mathcal{W}}_N^{\mathrm{B}} \times \widehat{\mathcal{M}}_N^{\mathrm{B}}$, that

$$\mathrm{iso}^{\mathcal{M},\iota}(wm) = \mathrm{iso}^{\mathcal{W},\iota}(w)\mathrm{iso}^{\mathcal{M},\iota}(m)$$

using the argument of (i) and Proposition-Definition 3.1.15.

□

Corollary 3.1.28. *Let $\iota \in \mathrm{Emb}(G)$ and $\phi \in \mathrm{Aut}(G)$. We have*

$$\mathrm{iso}^{\mathcal{M},\iota\circ\phi^{-1}} = \eta_\phi^{\mathcal{M}} \circ \mathrm{iso}^{\mathcal{M},\iota},$$

with $\eta_\phi^{\mathcal{M}} \in \mathrm{Aut}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}})$ given in Lemma 1.4.8.(ii).

Proof. It follows from Proposition 3.1.9 thanks to the commutativity of diagrams (3.34) and (1.39). ☐

3.2. The coproducts $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ and $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$.

3.2.1. *Comparison isomorphisms.*

Definition 3.2.1. For $(\iota, \lambda, \Psi) \in \mathrm{Emb}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define the topological $\mathbf{k}$-algebra-module isomorphism

$$\left({}^\Gamma\mathrm{comp}^{\mathcal{V},(1)}_{(\iota,\lambda,\Psi)}, {}^\Gamma\mathrm{comp}^{\mathcal{V},(10)}_{(\iota,\lambda,\Psi)}\right) : (\widehat{\mathcal{V}}_N^{\mathrm{B}}, \widehat{\mathcal{V}}_N^{\mathrm{B}}) \to (\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}}) \tag{3.36}$$

given by

$$\left({}^\Gamma\mathrm{comp}^{\mathcal{V},(1)}_{(\iota,\lambda,\Psi)}, {}^\Gamma\mathrm{comp}^{\mathcal{V},(10)}_{(\iota,\lambda,\Psi)}\right) := \left({}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}, {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}\right) \circ \left(\mathrm{iso}^{\mathcal{V},\iota}, \mathrm{iso}^{\mathcal{V},\iota}\right)$$

Proposition-Definition 3.2.2. *For $(\iota, \lambda, \Psi) \in \mathrm{Emb}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define the topological $\mathbf{k}$-algebra-module isomorphism*

$$\left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\lambda,\Psi)}, {}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}\right) : (\widehat{\mathcal{W}}_N^{\mathrm{B}}, \widehat{\mathcal{W}}_N^{\mathrm{B}}) \to (\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{W}}_G^{\mathrm{DR}}) \tag{3.37}$$

given by

$$\left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\lambda,\Psi)}, {}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}\right) := \left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}, {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right) \circ \left(\mathrm{iso}^{\mathcal{W},\iota}, \mathrm{iso}^{\mathcal{M},\iota}\right).$$

It is such that the following diagrams

$$\begin{array}{ccc} \widehat{\mathcal{W}}_N^{\mathrm{B}} & \xrightarrow{{}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\lambda,\Psi)}} & \widehat{\mathcal{W}}_N^{\mathrm{DR}} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{V}}_N^{\mathrm{B}} & \xrightarrow[{}^\Gamma\mathrm{comp}^{\mathcal{V},(1)}_{(\iota,\lambda,\Psi)}]{} & \widehat{\mathcal{V}}_N^{\mathrm{DR}} \end{array} \tag{3.38}$$

and

$$\begin{array}{ccc} \widehat{\mathcal{V}}_N^{\mathrm{B}} & \xrightarrow{{}^\Gamma\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\lambda,\Psi)}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{-\cdot 1_{\mathrm{B}}}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\ \widehat{\mathcal{M}}_N^{\mathrm{B}} & \xrightarrow[{}^\Gamma\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{3.39}$$

commute.

Proof. From Proposition-Definition 2.1.4 and Proposition 3.1.27.(ii), we have that the pairs $\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}, {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)$ and $\left(\mathrm{iso}^{\mathcal{W},\iota}, \mathrm{iso}^{\mathcal{W},\iota}\right)$ are isomorphisms in $\mathbf{k}$-alg-mod$_{\mathrm{top}}$; the composition is then an isomorphism in $\mathbf{k}$-alg-mod$_{\mathrm{top}}$. Next, the commutativity of the diagrams follows from the commutativity of Diagrams (2.9) and (3.19) and Diagrams (2.10) and (3.34). □

Recall the action of the group $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ given in Corollary 1.4.6. One has the following result:

Proposition 3.2.3. *For* $(\phi,\lambda,\Psi)\in\mathrm{Aut}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $(\iota,\nu,\Phi)\in\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, we have*

(i) ${}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)} = {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\circ{}^\Gamma\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\iota,\nu,\Phi)}$.

(ii) ${}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)} = {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\iota,\nu,\Phi)}$.

Proof.

(i) We have

$$\begin{aligned}{}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)} &= {}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota\circ\phi^{-1},\lambda\nu,\Psi\circledast\eta_\phi(\lambda\bullet\Phi))} = {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)\circledast(\nu,\eta_\phi(\Phi))}\circ\mathrm{iso}^{\mathcal{W},\iota\circ\phi^{-1}}\\ &= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\circ{}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\eta_\phi(\Phi))}\circ\eta^{\mathcal{W}}_\phi\circ\mathrm{iso}^{\mathcal{W},\iota}\\ &= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\circ\eta^{\mathcal{W}}_\phi\circ{}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\Phi)}\circ\mathrm{iso}^{\mathcal{W},\iota}\\ &= {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\circ{}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\nu,\Phi)},\end{aligned}$$

where the second equality follows from Lemma 1.4.3, the third equality from Corollary 2.1.7 and Corollary 3.1.16 and the fourth one from Corollary 2.1.11.(i).

(ii) We have

$$\begin{aligned}{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)} &= {}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota\circ\phi^{-1},\lambda\nu,\Psi\circledast\eta_\phi(\lambda\bullet\Phi))} = {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)\circledast(\nu,\eta_\phi(\Phi))}\circ\mathrm{iso}^{\mathcal{M},\iota\circ\phi^{-1}}\\ &= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\circ{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\nu,\eta_\phi(\Phi))}\circ\eta^{\mathcal{M}}_\phi\circ\mathrm{iso}^{\mathcal{M},\iota}\\ &= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\circ\eta^{\mathcal{M}}_\phi\circ{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\nu,\Phi)}\circ\mathrm{iso}^{\mathcal{M},\iota}\\ &= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)},\end{aligned}$$

where the second equality follows from Lemma 1.4.3, the third equality from Corollary 2.1.7 and Corollary 3.1.28 and the fourth one from Corollary 2.1.11.(ii). □

3.2.2. *The coproducts* $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N$ *and* $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N$.

Theorem 3.2.4. *The composition*

$$\left(\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\nu,\Phi)}\right)^{\otimes 2}\right)^{-1}, \left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)}\right)^{\otimes 2}\right)^{-1}\right)\circ\left(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G, \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\right)\circ\left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\nu,\Phi)}, {}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)}\right):$$
$$\left(\widehat{\mathcal{W}}^{\mathrm{B}}_N, \widehat{\mathcal{M}}^{\mathrm{B}}_N\right)\to\left((\widehat{\mathcal{W}}^{\mathrm{B}}_N)^{\otimes 2}, (\widehat{\mathcal{M}}^{\mathrm{B}}_N)^{\otimes 2}\right)$$

is independent of the choice of $(\iota,\nu,\Phi)\in\mathsf{DMR}_\times(\mathbf{k})$. *We denote it* $(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N,\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N)$. *Moreover, the pair* $(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N,\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N)$ *is an element of* $\mathrm{Cop}_{\mathbf{k}\text{-alg-mod}_{\mathrm{top}}}\left(\widehat{\mathcal{W}}^{\mathrm{B}}_N,\widehat{\mathcal{M}}^{\mathrm{B}}_N\right)$.

Proof. Let (ι,ν,Φ) and $(\iota',\nu',\Phi')\in\mathsf{DMR}_\times(\mathbf{k})$. Thanks to Proposition 1.4.14, there exists a unique $(\phi,\lambda,\Psi)\in(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}^G_0(\mathbf{k})$ such that $(\iota',\nu',\Phi')=(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)$. We have

$$\begin{aligned}&\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota',\nu',\Phi')}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota',\nu',\Phi')}\\
&=\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)\cdot(\iota,\nu,\Phi)}\\
&=\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)}\right)^{\otimes 2}\right)^{-1}\circ\left(\left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\circ{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)}\\
&=\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)},\end{aligned}$$

where the second equality comes from Proposition 3.2.3.(ii) and the last equality from the inclusion $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}^G_0(\mathbf{k})\subset\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k})$ of Corollary 2.2.5. Similary, we prove that

$$\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota',\nu',\Phi')}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G\circ{}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota',\nu',\Phi')}=\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\nu,\Phi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G\circ{}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\nu,\Phi)},$$

by replacing $\mathcal{M},(10)$ (resp. $\mathcal{M},\mathrm{DR}$) by $\mathcal{W},(1)$ (resp. $\mathcal{W},\mathrm{DR}$) in the exponents and the use of Proposition 3.2.3.(ii) by that of Proposition 3.2.3.(i); and using the the inclusion $(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathsf{DMR}^G_0(\mathbf{k})\subset\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k})$ of Corollary 2.2.5. Finally, $\left(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N,\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N\right)$ is an element of $\mathrm{Cop}_{\mathbf{k}\text{-alg-mod}_{\mathrm{top}}}\left(\widehat{\mathcal{W}}^{\mathrm{B}}_N,\widehat{\mathcal{M}}^{\mathrm{B}}_N\right)$ since the pair $\left(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G,\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\right)$ is an element of $\mathrm{Cop}_{\mathbf{k}\text{-alg-mod}_{\mathrm{top}}}\left(\widehat{\mathcal{W}}^{\mathrm{DR}}_G,\widehat{\mathcal{M}}^{\mathrm{DR}}_G\right)$ thanks to Lemma 1.1.3 and the pair $\left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\lambda,\Psi)},{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}\right)$ is a $\mathbf{k}$-algebra-module isomorphism thanks to Proposition-Definition 3.2.2. □

Corollary 3.2.5. *We have* $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N(1_\mathrm{B})=1_\mathrm{B}^{\otimes 2}$.

Proof. From Theorem 3.2.4, let us compute $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N(1_\mathrm{B})$ by considering an element $(\iota,\lambda,\Psi)\in\mathsf{DMR}_\times(\mathbf{k})$. First, we have

$$(3.40)\qquad {}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}(1_\mathrm{B})={}^\Gamma\mathrm{comp}^{\mathcal{V},(10)}_{(\iota,\lambda,\Psi)}(1)\cdot 1_{\mathrm{DR}}=\Gamma_\Psi^{-1}(-e_1)\beta(\Psi\otimes 1)\cdot 1_{\mathrm{DR}}=\Psi^\star.$$

Therefore,

$$\begin{aligned}\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N(1_\mathrm{B})&=\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}(1_\mathrm{B})\\
&=\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G(\Psi^\star)\\
&=\left(\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1}(\Psi^\star\otimes\Psi^\star)=1_\mathrm{B}^{\otimes 2},\end{aligned}$$

where the first and last equalities come from (3.40) and the second one from the fact that $\Psi \in \mathsf{DMR}^{\iota}_{\lambda}(\mathbf{k})$. □

Corollary 3.2.6.

(i) The pair $(\widehat{\mathcal{W}}^{\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N)$ *is an object in the category* $\mathbf{k}$-Hopf$_{\mathrm{top}}$.

(ii) The pair $(\widehat{\mathcal{M}}^{\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N)$ *is an object in the category* $\mathbf{k}$-coalg$_{\mathrm{top}}$.

(iii) The pair $\left((\widehat{\mathcal{W}}^{\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N), (\widehat{\mathcal{M}}^{\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N)\right)$ *is an object in the category* $\mathbf{k}$-HAMC$_{\mathrm{top}}$.

Proof.

(i) From Theorem 3.2.4, it follows that $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N$ is an algebra morphism. In addition, one checks that the coassociativity of $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N$ follows from the coassociativity of $\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G$.

(ii) From Theorem 3.2.4, it follows that $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N$ is a $\mathbf{k}$-module morphism. In addition, one checks that the coassociativity of $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N$ follows from the coassociativity of $\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G$.

(iii) It follows from (i) and (ii) and the fact that the pair $(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N)$ is an element of $\mathrm{Cop}_{\mathbf{k}\text{-alg-mod}_{\mathrm{top}}}\left(\widehat{\mathcal{W}}^{\mathrm{B}}_N, \widehat{\mathcal{M}}^{\mathrm{B}}_N\right)$.

□

4. Expression of the torsor $\mathsf{DMR}_\times(\mathbf{k})$ in terms of the Betti and de Rham coproducts

In this section, we show that $\mathsf{DMR}_\times(\mathbf{k})$ is a subtorsor of a stabilizer torsor of the pair of coproducts $\left(\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\right)$. In §4.1, we define the setwise stabilizers $\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\right)(\mathbf{k})$ and $\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G\right)(\mathbf{k})$ and show that they are equipped with a torsor structure for the actions of the stabilizer groups $\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\right)(\mathbf{k})$ and $\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G\right)(\mathbf{k})$ respectively. In §4.2.1, we obtain a chain of inclusions of torsors involving these stabilizers and $\mathsf{DMR}_\times(\mathbf{k})$.

4.1. The stabilizer subtorsors.

Definition 4.1.1.

(i) We denote $\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G\right)(\mathbf{k})$ the setwise stabilizer of the pair of coproducts $\left(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G\right) \in \mathrm{Cop}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{W}}^{\mathrm{B}}_N) \times \mathrm{Cop}_{\mathbf{k}\text{-alg}_{\mathrm{top}}}(\widehat{\mathcal{W}}^{\mathrm{DR}}_G)$ given by

$$\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G\right)(\mathbf{k}) :=$$
$$\left\{(\iota,\nu,\Phi) \in \mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \,|\, \left({}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\nu,\Phi)}\right)^{\otimes 2} \circ \widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N = \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G \circ {}^\Gamma\mathrm{comp}^{\mathcal{W},(1)}_{(\iota,\nu,\Phi)}\right\}.$$

(ii) We denote $\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\right)(\mathbf{k})$ the setwise stabilizer of the pair of coproducts $\left(\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N, \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G\right) \in \mathrm{Cop}_{\mathbf{k}\text{-mod}_{\mathrm{top}}}(\widehat{\mathcal{M}}^{\mathrm{B}}_N) \times \mathrm{Cop}_{\mathbf{k}\text{-mod}_{\mathrm{top}}}(\widehat{\mathcal{W}}^{\mathrm{DR}}_G)$

given by

$$\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}},\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k}) :=$$
$$\left\{(\iota,\nu,\Phi)\in\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)\,|\,\left({}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)}\right)^{\otimes 2}\circ\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}=\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\circ{}^\Gamma\mathrm{comp}^{\mathcal{M},(10)}_{(\iota,\nu,\Phi)}\right\}.$$

Remark 4.1.2. Theorem 3.2.4 implies that $\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}},\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k})$ and $\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}},\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k})$ contain $\mathsf{DMR}_\times(\mathbf{k})$, which implies that these are nonempty sets.

Proposition 4.1.3.

i. The pair

$$\left(\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k}),\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}},\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k})\right)$$

is a subtorsor of $\Big((\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)\Big)$.

ii. The pair

$$\left(\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k}),\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}},\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k})\right)$$

is a subtorsor of $\Big((\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)\Big)$.

In order the prove this, we will need the following Lemma:

Lemma 4.1.4 ([EF2, Lemma 2.6])**.** *Let (H,T) be a torsor, and let V,V' be $\mathbf{k}$-modules. Let $\rho:H\to\mathrm{Aut}_{\mathbf{k}\text{-mod}}(V)$ be a group morphism and let $\rho':T\to\mathrm{Iso}_{\mathbf{k}\text{-mod}}(V',V)$ be a map such that for any $h\in H$, $x\in T$, one has $\rho'(h\cdot x)=\rho(h)\circ\rho'(x)$. Let $v\in V$ and $v'\in V'$. Then $\mathrm{Stab}_H(v):=\{h\in H\,|\,\rho(h)(v)=v\}$ is a subgroup of H, and either $\mathrm{Stab}_T(v,v'):=\{x\in T\,|\,\rho'(v')=v\}$ is empty, or $(\mathrm{Stab}_H(v),\mathrm{Stab}_T(v,v'))$ is a subtorsor of (H,T).*

Proof of Proposition 4.1.3. It follows from Lemma 4.1.4 by setting :

- $(H,T)=((\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle),\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle))$;
- $V=\mathrm{Cop}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}})$ (resp. $V=\mathrm{Cop}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}})$);
- $V'=\mathrm{Cop}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{W}}_N^{\mathrm{B}})$ (resp. $V'=\mathrm{Cop}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_N^{\mathrm{B}})$);
- $v=\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ (resp. $v=\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$);
- $v'=\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ (resp. $v'=\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$);
- $\rho:(\phi,\lambda,\Psi)\mapsto\left(V\ni D_{\mathrm{DR}}^{\mathcal{W}}\mapsto\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\right)^{\otimes 2}\circ D_{\mathrm{DR}}^{\mathcal{W}}\circ\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\phi,\lambda,\Psi)}\right)^{-1}\in V\right)$ (resp. $\rho:(\phi,\lambda,\Psi)\mapsto\left(V\ni D_{\mathrm{DR}}^{\mathcal{M}}\mapsto\left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)^{\otimes 2}\circ D_{\mathrm{DR}}^{\mathcal{M}}\circ\left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\phi,\lambda,\Psi)}\right)^{-1}\in V\right)$);

- $\rho' : (\phi, \lambda, \Psi) \mapsto \left(V' \ni D_{\mathrm{B}}^{\mathcal{W}} \mapsto \left({}^{\Gamma}\mathrm{comp}_{(\phi,\lambda,\Psi)}^{\mathcal{W},(1)} \right)^{\otimes 2} \circ D_{\mathrm{B}}^{\mathcal{W}} \circ \left({}^{\Gamma}\mathrm{comp}_{(\phi,\lambda,\Psi)}^{\mathcal{W},(1)} \right)^{-1} \in V \right)$ (resp. $\rho' : (\phi, \lambda, \Psi) \mapsto \left(V' \ni D_{\mathrm{B}}^{\mathcal{M}} \mapsto \left({}^{\Gamma}\mathrm{comp}_{(\phi,\lambda,\Psi)}^{\mathcal{M},(10)} \right)^{\otimes 2} \circ D_{\mathrm{B}}^{\mathcal{M}} \circ \left({}^{\Gamma}\mathrm{comp}_{(\phi,\lambda,\Psi)}^{\mathcal{M},(10)} \right)^{-1} \in V \right)$).

Finally, for $(\phi, \lambda, \Psi) \in H$ and $(\iota, \nu, \Phi) \in T$, the identity

$$\rho((\phi, \lambda, \Psi) \cdot (\iota, \nu, \Phi)) = \rho(\phi, \lambda, \Psi) \circ \rho'(\iota, \nu, \Phi)$$

follows from Proposition 3.2.3. □

4.2. Inclusion of stabilizer torsors.

Theorem 4.2.1. *We have the following inclusions of torsors*

$$\begin{gathered}
\left((\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathsf{DMR}_0^G(\mathbf{k}), \mathsf{DMR}_\times(\mathbf{k})\right) \\
\cap \\
\left(\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k}), \mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k})\right) \\
\cap \\
\left(\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k}), \mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k})\right) \\
\cap \\
\left((\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \mathrm{Emb}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)\right)
\end{gathered}$$

In order the prove this, we will need the following Lemmas:

Lemma 4.2.2 ([EF2, Lemma 2.3])**.** *Let (H, T) be a torsor and let (H', T') and (H'', T'') be subtorsors of (H, T) such that $T' \cap T'' \neq \varnothing$. Then $(H' \cap H'', T' \cap T'')$ is a subtorsor of both (H', T') and (H'', T''), therefore of (H, T).*

Lemma 4.2.3 ([EF2, Lemma 2.7])**.** *Let (H, T) be a torsor and let (H_0, T_0) and (H_1, T_1) be subtorsors of (H, T) such that $T_0 \subset T_1$. Then (H_0, T_0) is a subtorsor of (H_1, T_1). If, moreover, $H_0 = H_1$ then the subtorsors (H_0, T_0) and (H_1, T_1) are equal.*

Proof of Theorem 4.2.1. The group-part inclusion is shown in Corollary 2.2.5. The first and last set-part inclusions are immediate. It remains to show that

$$\mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k}) \subset \mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k}).$$

In Lemmas 4.2.2 and 4.2.3, set

$$(H, T) = \left((\mathrm{Aut}(G) \times \mathbf{k}^\times) \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \mathrm{Emb}(G) \times \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)\right).$$

First, let us apply Lemma 4.2.2 for

- $(H', T') = \left(\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k}), \mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)(\mathbf{k})\right)$;
- $(H'', T'') = \left(\mathsf{Stab}_{(\mathrm{Aut}(G)\times\mathbf{k}^\times)\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k}), \mathsf{Stab}_{\mathrm{Emb}(G)\times\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)}\left(\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right)(\mathbf{k})\right)$.

From Remark 4.1.2, we have that $T' \cap T'' \neq \varnothing$. Therefore, $(H' \cap H'', T' \cap T'')$ is a subtorsor of (H'', T''). Second, let us apply Lemma 4.2.3 for

- $(H_0, T_0) = (H' \cap H'', T' \cap T'')$;
- $(H_1, T_1) = (H', T')$.

We have that $T_0 = T' \cap T'' \subset T' = T_1$. In addition,

$$H_0 = H' \cap H'' = H' = H_1,$$

where the second equality follows from the stabilizer group inclusion in Corollary 2.2.5. Finally, it follows that $T' \cap T'' = T_0 = T_1 = T'$. Thus $T' \subset T''$, which is the wanted inclusion of setwise stabilizers. □

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Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg, 7 rue René Descartes, 67000 Strasbourg, France

Email address: kyaddaden@math.unistra.fr